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Broué's conjecture for non-principal 3-blocks of finite groups

Shigeo Koshitani[a,*], Naoko Kunugi[b], Katsushi Waki[c]

[a] *Department of Mathematics, Faculty of Science, Chiba University, Yayoi-cho, Inage-Ku, Chiba-shi 263-8522, Japan*

[b] *Department of Mathematics, Faculty of Science, Ochanomizu University, Bunkyo-ku, Tokyo 112-8610, Japan*

[c] *Department of Mathematical System Science, Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan*

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Abstract

In representation theory of finite groups, there is a well-known and important conjecture due to M. Broué. He conjectures that, for any prime p, if a p-block A of a finite group G has an abelian defect group D, then A and its Brauer correspondent p-block B of $N_G(D)$ are derived equivalent. We demonstrate in this paper that Broué's conjecture holds for two non-principal 3-blocks A with elementary abelian defect group D of order 9 of the O'Nan simple group and the Higman–Sims simple group. Moreover, we determine these two non-principal block algebras over a splitting field of characteristic 3 up to Morita equivalence. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction and notation

In representation theory of finite groups, one of the most important problems is to give an affirmative answer, if it is true, to a conjecture introduced by Broué [4]. He actually conjectures the following:

* Corresponding author.

E-mail addresses: koshitan@math.s.chiba-u.ac.jp (S. Koshitani), mkunugi@g.math.s.chiba-u.ac.jp (N. Kunugi), slwaki@cc.hirosaki-u.ac.jp (K. Waki).

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0.1. Conjecture (Broué [4, 6.2. Question], see König and Zimmermann [19, Conjecture in p. 132]).

For a prime p, let A be a p-block of a finite group G with defect group D, and let B be a p-block of $N_G(D)$ such that B is the Brauer correspondent of A. Then, A and B would be derived equivalent provided D is abelian.

There are only few special cases where Broué's conjecture (0.1) has been checked even in the case that A is the principal p-block, see [5, p. 15; 19, p. 136]. For non-principal p-blocks, Conjecture (0.1) is considered only in the cases that D is cyclic in a paper by Rickard [32], that G is a p-solvable group in a paper by Harris–Linckelmann [9], and that D is the Klein four group $C_2 \times C_2$ in [33]. The purpose of this article is to make observations for non-principal 3-blocks A of two sporadic simple groups, the O'Nan and the Higman–Sims simple groups, with elementary abelian defect group D of order 9. Namely, our main result is the following:

0.2. Theorem. *Let $(\mathscr{K}, \mathcal{O}, k)$ be a splitting 3-modular system for all subgroups of a finite group G (see below for the definition of a modular system).*

(i) *Assume that $G = O'N$, the O'Nan simple group. If $\mathcal{O}A = \hat{A}$ is the non-principal block algebra of $\mathcal{O}G$ with elementary abelian defect group D of order 9 and if $\mathcal{O}B = \hat{B}$ is the Brauer correspondent of $\mathcal{O}A$ in $N_G(D)$, then $\mathcal{O}A$ and $\mathcal{O}B$ are splendidly equivalent.*

(ii) (*see Holm* [11, p. 60]) *The same as in* (i) *holds for $G = HS$, the Higman–Sims simple group.*

It is announced in [11] by Holm that he himself has proved (0.2)(ii) (probably over k instead of $\mathcal{O}$). Since he has not written any proof of it, it may be useful to give a whole proof in this article. Furthermore, as a corollary of (0.2)(i), we also get the following:

0.3. Corollary. *Keep the notation in* (0.2)(i). *Let q be a power of a prime such that $q \equiv 4$ or $7 \pmod 9$. Then, all the principal block algebras $B_0(\mathcal{O}[L_3(q)])$ of the projective special linear groups $L_3(q)$ are splendidly Morita* (*hence Puig*) *equivalent to the non-principal block algebra $\mathcal{O}A$ of $\mathcal{O}[O'N]$, and the block algebra A over k is completely determined up to Morita equivalence.*

Moreover, by a few more calculations after getting (0.2)(ii), we can determine the structure of the block algebra A as k-algebra, completely up to Morita equivalence. In her thesis [10] Hicks almost determines A up to Morita equivalence. However, there is one unspecified β_1 there [10, p. 133]. We can actually determine β_1 such that $\beta_1 = -1$ in this article without using [10]. That is,

0.4. Theorem. *The non-principal block algebra A of $k[HS]$ of the Higman–Sims simple group HS is completely determined up to Morita equivalence* (*see* (5.10) *Theorem*).

The strategy of, for instance, (0.2)(i) is the following. Let A be the non-principal block algebra of kG with defect group $D = C_3 \times C_3$, where $G = O'N$, and let B be its Brauer correspondent in $k[N_G(D)]$. Then, B is Morita equivalent to $B' = k[D : Q_8]$, which is the Brauer correspondent of the principal block algebra A' of kG' where $G' = L_3(4)$. Then, each Green correspondent $f(S_i)$ of a simple kG-module S_i in A corresponds to a Green correspondent $f'(S_i')$ of a simple kG'-module S_i' in A' via the above Morita equivalence. Hence, we can apply Okuyama's method to A and B, in parallel with the steps he takes in applying it to A' and B' in [28, Example 4.6]. Hence, we finally know that A and B are derived (even splendidly) equivalent and furthermore that A and A' are Morita (even splendidly Morita) equivalent by passing through the Morita equivalent block algebras B and B'.

Throughout this article we use the following notation and terminology. Let G be a finite group and p a prime. Let $(\mathscr{K}, \mathscr{O}, k)$ be a splitting p-modular system for all subgroups of G, namely, $\mathscr{O}$ is a discrete valuation ring of rank one with unique maximal ideal $J(\mathscr{O}) = (\pi)$ for $\pi \in \mathscr{O}$, $\mathscr{K}$ is the quotient field of $\mathscr{O}$ with characteristic zero and k is its residue field $\mathscr{O}/(\pi)$ with characteristic p. Here, a module always means a finitely generated right module, unless stated otherwise. We mean by an $\mathscr{O}G$-lattice a right $\mathscr{O}G$-module which is $\mathscr{O}$-free of finite rank. When a kG-module M uniquely lifts to an $\mathscr{O}G$-lattice, we denote the $\mathscr{O}G$-lattice by $\hat{M}$. If $\hat{M}$ is an $\mathscr{O}G$-lattice, we write $\chi_{\hat{M}}$ for the ordinary character (namely, $\mathscr{K}$-character) of G afforded by the right $\mathscr{K}G$-module $\hat{M} \otimes_{\mathscr{O}} \mathscr{K}$. In particular, for a projective kG-module P, we sometimes write Φ_P for the $\mathscr{K}$-character of G afforded by $\hat{P}$, instead of $\chi_{\hat{P}}$. When D is a p-subgroup of G, we use a term *a Green correspondence* with respect to (G, D, H) following [26, Chapter 4, Section 4].

Let A be a finite dimensional k-algebra, and let M and N be A-modules. We denote by 1_A, $J(A)$ and $\ell(A)$, respectively, the unit element of A, the Jacobson radical of A and the number of all non-isomorphic simple A-modules. We denote by mod-A the category of finitely generated right A-modules. We write $P(M)$ for the projective cover of M, ΩM for the kernel of the projective cover $P(M) \twoheadrightarrow M$, that is, Ω is the Heller operator, and $\mathrm{Soc}(M)$ for the socle of M. For a positive integer i, we define $\mathrm{rad}^i(M)$ by $M \cdot J(A)^i$ and $\mathrm{Soc}_i(M)$ by $\mathrm{Soc}_1(M) = \mathrm{Soc}(M)$ and $\mathrm{Soc}_i(M)/\mathrm{Soc}_{i-1}(M) = \mathrm{Soc}(M/\mathrm{Soc}_{i-1}(M))$ if $i \geqslant 2$. We write $j(M)$ for the Loewy (and hence socle) length of M, namely, $j(M)$ is the least positive integer j such that $M \cdot J(A)^j = 0$, equivalently, $\mathrm{Soc}_j(M) = M$. The ith Loewy layer $M \cdot J(A)^{i-1}/M \cdot J(A)^i$ is denoted by $L_i(M)$ for $i = 1, \ldots, j(M)$. Let $S_1, \ldots, S_n$ be simple A-modules, possibly $S_i \cong S_j$ for a pair (i, j) with $i \neq j$. Then, we write $U(S_1, \ldots, S_n)$ for a uniserial A-module M such that $L_i(M) \cong S_i$ for $i = 1, \ldots, n$. Note that $U(S_1, \ldots, S_n)$ is not necessarily determined up to isomorphism in general. We write $N|M$ if N is a direct summand of M as A-module. We write $M = S_1 + \cdots + S_n$, *as composition factors* when $\{S_1, \ldots, S_n\}$ is the set of all composition factors of M (possibly $S_i \cong S_j$ for $i \neq j$). The multiplicity of S_i in the composition factors of M is denoted by $c_M(S_i)$. The Cartan invariant with respect to S_i and S_j is denoted by $c(S_i, S_j)$ or simply by c_{S_i, S_j}, namely, $c(S_i, S_j) = c_{P(S_i)}(S_j)$. We say that M has a filtration

$$M = \begin{array}{ccc} N_1 & & \\ & \left| N_2 \right| & \\ & & N_3 \end{array}$$

for A-modules N_1, N_2 and N_3 when M has submodules M_1, M_2 and M_3 satisfying $M \supseteq M_1 \supseteq M_2 \supseteq M_3$, $M/M_1 \cong N_1$, $M_1/M_2 \cong N_2$ and $M_3 \cong N_3$. We write that $M = N \oplus$ (proj) if M is a direct sum of N and a projective A-module.

Let n be a positive integer. We denote by C_n, D_n, Q_n and SD_n, respectively, the cyclic group, the dihedral group, the generalized quaternion group and the semi-dihedral group of order n. We write Σ_n and A_n for the symmetric and the alternating group on n letters. We write $L_3(q)$ for $\mathrm{PSL}_3(q)$ when q is a power of a prime. The Mathieu group of degree n is denoted by M_n. For a prime p and a non-negative integer r we write $v_p(n) = r$ when $p^r | n$ and $p^{r+1} \nmid n$. For a p-block A of G, $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$, respectively, denote the sets of all irreducible ordinary and Brauer characters of G belonging to A, and we write $k(A)$ and $\ell(A)$, respectively, for the numbers of elements of the sets $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$. We write $G = H : K$ when G is a semi-direct product of H by K such that $H \lhd G$. We write $\Delta(G)$ for the diagonal subgroup $\{(g,g) \in G \times G \mid g \in G\}$ of $G \times G$. The trivial kG-module is denoted by k_G. We write $B_0(kG)$ for the principal block algebra of the group algebra kG.

Let M and N be kG-modules. Then, $\mathrm{Hom}_{kG}(M,N)$ and $\dim_k[\mathrm{Hom}_{kG}(M,N)]$ are denoted by $(M,N)^G$ and $[M,N]^G$, respectively. We denote by $\underline{\mathrm{Hom}}_{kG}(M,N)$ the k-vector space $(M,N)^G/\{\text{all projective hom's}\}$. We write M^* for $\mathrm{Hom}_k(M,k)$, the k-dual of M, which is also a right kG-module as usual. For a projective indecomposable kG-module P, $[P|M]$ denotes the multiplicity of P in M as direct summands. For a subgroup H of G, we denote by $P_H(M)$ the (relatively) H-projective cover of M, and by $\Omega_H(M)$ the kernel of the epimorphism $P_H(M) \twoheadrightarrow M$, namely Ω_H is the (relatively) H-projective Heller operator (see [17, p. 29]). Now, $M \otimes_k N$ becomes a right kG-module via diagonal action $(m \otimes n)g = mg \otimes ng$ for $m \in M$, $n \in N$ and $g \in G$. On the other hand, since $G \cong \Delta G$ via $g \mapsto (g,g)$ for $g \in G$, N is considered as right $k[\Delta G]$-module as well via $n \cdot (g,g) = ng$ for $n \in N$ and $g \in G$. Let Z be a right $k[G \times G]$-module. Then, as usual, Z may be considered as (kG,kG)-bimodule via $g_1 \cdot z \cdot g_2 = z(g_1^{-1}, g_2)$ for $g_i \in G$ and $z \in Z$. For ordinary characters χ and ψ of G, $(\chi,\psi)_G$ denotes the inner product of χ and ψ in G. For other notation and terminology, see the book of Nagao and Tsushima [26].

1. Lemmas

In this section we list several lemmas which are useful to get our main result. Throughout this section we fix a finite group G.

1.1. Lemma (Scott; see Landrock [22, II, Theorem 12.4, I, Proposition 14.8]). (i) *If M is a trivial source kG-module, then M uniquely lifts to a trivial source $\mathcal{O}G$-lattice $\hat{M}$.*

(ii) *If M and N are both trivial source kG-modules, then* $[M,N]^G = (\chi_{\hat{M}}, \chi_{\hat{N}})_G$.

1.2. Lemma (Green–Landrock–Scott; see Landrock [22, II, Lemma 12.6]). *Let M be a trivial source kG-module, so that M uniquely lifts to a trivial source $\mathcal{O}G$-lattice $\hat{M}$* (*see* (1.1)(i)).

(i) *Let Q be a p-subgroup of G. Then*, $\dim_k[(\mathrm{Soc}(M\downarrow_Q)] = (\chi_{\hat{M}}\downarrow_Q, 1_Q)_Q$.

(ii) *Let x be a p-element in G. Then $\chi_{\hat{M}}(x)$ equals the number of indecomposable direct summands of the $k\langle x\rangle$-module $M\downarrow_{\langle x\rangle}$ which are isomorphic to the trivial $k\langle x\rangle$-module $k_{\langle x\rangle}$. In particular, $\chi_{\hat{M}}(x)$ is a non-negative integer.*

(iii) *Let x be a p-element in G. Then, $\chi_{\hat{M}}(x) > 0$ if and only if x belongs to some vertex of M.*

1.3. Lemma (Zassenhaus and others; see Landrock [22, I, Theorem 17.3]). *Let $\hat{M}$ be an $\mathcal{O}G$-lattice. Assume that $\chi_{\hat{M}} = \chi_1 + \chi_2$ where each χ_i is an ordinary character of G. Then, $\hat{M}$ contains an $\mathcal{O}$-pure $\mathcal{O}G$-submodule $\hat{N}_i$ such that $\chi_i = \chi_{\hat{N}_i}$ for each i. Let $M = \hat{M}/\hat{M}\pi$ and $N_i = \hat{N}_i/\hat{N}_i\pi$ for each i. Then, M contains kG-submodules M_1 and M_2 with $M_i \cong N_i$ for $i = 1, 2$.*

1.4. Lemma (Thompson; see Landrock [22, I, Corollary 17.4]). *Let S be a simple kG-module, and assume that $\Phi_{P(S)} = \chi_1 + \chi_2$ for ordinary characters χ_i of G. Then, there exists an $\mathcal{O}G$-lattice $\hat{M}_1$ such that $\chi_{\hat{M}_1} = \chi_1$ and* $\mathrm{Soc}(\hat{M}_1/\hat{M}_1\pi) \cong S$.

1.5. Lemma (Landrock; see Landrock [22, I, Lemma 9.10]). *Let S and T be simple kG-modules. Then, $[L_i(P(S)), T]^G = [L_i(P(T^*)), S^*]^G$ for all $i = 1, 2, \ldots$.*

1.6. Lemma (Robinson [34, Theorem 3]). *Let H be a subgroup of G, and let S and T, respectively, be a simple kG-module and a simple kH-module. Then, $[P(S) \mid T\uparrow^G] = [P(T) \mid S\downarrow_H]$.*

1.7. Lemma (Brauer–Conlon–Green–Nagao; see Nagao and Tsushima [26, Chapter 5, Theorems 3.8, 3.10, Corollary 3.11, Theorem 3.12]). *Let $R \in \{k, \mathcal{O}\}$. Let D be a p-subgroup of G, and let $H = N_G(D)$. Assume that A and B, respectively, are block algebras of RG and RH with defect group D such that A and B correspond each other via the Brauer correspondence, namely, $A = B^G$ (block induction). Let $f: G \to H$ and $g: H \to G$ be the Green correspondences with respect to (G, D, H). Let $\mathfrak{Z} = \{Q \mid Q \leqslant D,\ Q \nleqslant D^x \cap D$ for all $x \in G - H\}$. Then we get the following*:

(i) *If X is an indecomposable RG-module in A such that a vertex of X is in $\mathfrak{Z}$, then its Green correspondent fX of X belongs to B.*

(ii) *If Y is an indecomposable RH-module in B such that a vertex of Y is in $\mathfrak{Z}$, then its Green correspondent gY of Y belongs to A.*

1.8. Lemma (Green; see Nagao and Tsushima [26, Chapter 5, Theorem 1.9]). *Let M be an indecomposable kG-module in a block algebra A of kG with defect group D. If $v_p(\dim_k M) = v_p(|G|) - v_p(|D|)$, then M has D as its vertex.*

1.9. Lemma (Erdmann–Kawata; see Erdmann [7, Theorem 1] and Kawata [16, Theorem 1.5]). *Let A be a block algebra of kG which is of wild representation type. Suppose, moreover, that there does not exist any simple kG-module S in A such that the Cartan invariants of S satisfy $c(S, S) = 2$; and $c(S, S') = 1$ or 0 for every simple*

kG-module S' in A with $S' \not\cong S$. Then, for any simple kG-module T in A, the heart $\mathscr{H}(P(T)) = [P(T) \cdot J(kG)]/T$ of $P(T)$ is an indecomposable kG-module.

Proof. Suppose that there is a simple kG-module T in A such that $\mathscr{H}(P(T))$ is decomposable. Let Θ be a connected component of the stable Auslander–Reiten quiver of A containing T. Since A is of wild representation type, a theorem of Erdmann [7, Theorem 1] shows that the tree class of Θ is of A_∞-type. Thus, as remarked in [16, ℓ.11 of Introduction], T does not lie at the end of Θ. Therefore, a theorem of Kawata [16, Theorem 1.5] implies that there exists a simple kG-module S in A such that $c(S,S) = 2$ and $c(S,S') = 0$ or 1 for any simple kG-module $S' \not\cong S$ in A, a contradiction. □

1.10. Lemma. *Let U and V be right kG-modules. Then, a map*

$$\phi : U \otimes_{kG} (V_{\Delta G} \uparrow^{G\times G}) \to U \otimes_k V$$

defined by

$$u \otimes_{kG} (v \otimes_{k[\Delta G]} (g_1, g_2)) \mapsto ug_1^{-1}g_2 \otimes vg_2$$

for $u \in U$, $v \in V$ and $g_1, g_2 \in G$ is an isomorphism of right kG-modules.

Proof. First of all, we claim that, for $u \in U$, $v \in V$ and $g_1, g_2 \in G$,

$$u \otimes_{kG} (v \otimes_{k[\Delta G]} (g_1, g_2)) = ug_1^{-1}g_2 \otimes_{kG} (v \cdot (g_2, g_2) \otimes_{k[\Delta G]} (1,1)) \qquad (*)$$

in $U \otimes_{kG} (V \otimes_{k[\Delta G]} k[G \times G])$. We know that ϕ is a right kG-module-homomorphism, and that the inverse map of ϕ is given by

$$\psi : U \otimes_k V \to U \otimes_{kG} (V_{\Delta G} \uparrow^{G\times G}) = U \otimes_{kG} (V \otimes_{k[\Delta G]} k[G \times G])$$

such that

$$u \otimes v \mapsto u \otimes_{kG} v \otimes_{k[\Delta G]} (1,1). \qquad □$$

2. Preliminaries for 3-modular representations of the O'Nan simple group

From now on except in Section 5 we fix notation, and we introduce several fundamental results (lemmas) on 3-modular representations of the O'Nan simple group.

2.1. Notation. We assume that G is the O'Nan simple group, namely $G = O'N$, $|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$, and $|\mathrm{Out}(G)| = 2$, so that we fix an element $\sigma \in \mathrm{Aut}(G) - G$ (see [30] or [6, pp. 132–133]). Note that Sylow 3-subgroups of G are elementary abelian of order 3^4 (see [30, p. 422, ℓ. 23]).

2.2. Lemma. *The O'Nan simple group G has eight 3-blocks, namely, the principal 3-block $B_0(G)$, a non-principal 3-block A with elementary abelian defect group D of order 9, and six 3-blocks of defect zero.*

Proof. see Jansen and Wilson [15, Section 3, pp. 73–75; 4, A2.2, p. 88]. □

2.3. Notation. We use the notation A and D as in (2.2). We denote $J(kG)$ by J.

2.4. Lemma. (i) *It holds that* $k(A)=6$, *and we can write* $\mathrm{Irr}(A)=\{\chi_{10944},\chi_{52668},\chi_{58311_1},\chi_{58311_2},\chi_{58311_3},\chi_{58653}\}$.

(ii) *Each* $\chi_i\in\mathrm{Irr}(A)$ *has a value on* $3A$ *as the following, where* $3A$ *is a unique conjugacy class of* G *with elements of order* 3.

In [6, p. 133]	χ_2	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
Here	χ_{10944}	χ_{52668}	χ_{58311_1}	χ_{58311_2}	χ_{58311_3}	χ_{58653}
$3A$	9	18	-9	-9	-9	9

(iii) *Moreover*, $\sigma\in\mathrm{Aut}(G)-G$ *acts on the set* $\mathrm{Irr}(A)$ *such that* $\chi_{58311_2}^{\sigma}=\chi_{58311_3}$ *and* $\chi^{\sigma}=\chi$ *for all* $\chi\in\mathrm{Irr}(A)-\{\chi_{58311_2},\chi_{58311_3}\}$.

Proof. This follows from [15, Section 3, pp. 74–75; 6, p. 133] (see [4, p. 88]). □

2.5. Notation. We use the notation χ_{10944}, χ_{52668}, χ_{58311_1}, χ_{58311_2}, χ_{58311_3}, χ_{58653} as in (2.4). Note that the subindices of χ_i's mean the degrees.

2.6. Lemma. (i) *It holds that* $\ell(A)=5$, *and we can write* $\mathrm{IBr}(A)=\{S,T,S_1,S_2,S_3\}$, *namely*, S, T, S_1, S_2 *and* S_3 *are all non-isomorphic simple kG-modules in A such that*

In [15, p. 73]	φ_{13}	φ_{18}	φ_7	φ_8	φ_9
Here	S	T	S_1	S_2	S_3
k-dim	10 944	41 724	5643	5643	5643

(ii) *All simple kG-modules in A are self-dual, and have D as their vertices.*

(iii) $\sigma\in\mathrm{Aut}(G)-G$ *acts on the set* $\mathrm{IBr}(A)$ *such that* $S_2^{\sigma}=S_3$ *and* $S'^{\sigma}=S'$ *for all* $S'\in\mathrm{IBr}(A)-\{S_2,S_3\}$.

(iv) *The* 3-*decomposition matrix of A has the following form*:

	S	T	S_1	S_2	S_3
χ_{10944}	1	.	.	.	.
χ_{52668}	1	1	.	.	.
χ_{58311_1}	1	1	1	.	.
χ_{58311_2}	1	1	.	1	.
χ_{58311_3}	1	1	.	.	1
χ_{58653}	.	1	1	1	1

(v) *The Cartan matrix of A has the following form*:

	$P(S)$	$P(T)$	$P(S_1)$	$P(S_2)$	$P(S_3)$
S	5	4	1	1	1
T	4	5	2	2	2
S_1	1	2	2	1	1
S_2	1	2	1	2	1
S_3	1	2	1	1	2

Proof. Follows by Jansen and Wilson [15, p. 75, Table 6; p. 73, Table 2], Knörr [18, 3.7 Corollary(i)] and Conway et al. [6, p. 133]. □

2.7. Lemma. (i) *Let $H = N_G(D)$. Then, $H = D:(4 \times A_6) \cdot 2 = [(D:4) \times A_6] \cdot 2$, and H is a maximal subgroup of G. Furthermore, $C_G(D) = D \times A_6$ and $H/C_G(D) \cong Q_8$.*

(ii) *σ in Notation (2.1) can be chosen as an element in $H \cdot 2$ which is a maximal subgroup of* $\mathrm{Aut}(G) = G.2$. *Namely, $\sigma \downarrow_H \in \mathrm{Aut}(H) - \mathrm{Inn}(H)$.*

(iii) *D is a T.I. set in G.*

Proof. (i) By (2.2), the inertial quotient of A is Q_8. Then, by Conway et al. [6, p. 132] (or [40,41]), we get the first part. Hence, we may assume that D is the group R in [30, p. 446]. Then, $C_G(D) = D \times A_6$ by O'Nan [30, Lemmas 5.3 and 5.5] (see [4, p. 88]).

(ii) We get this by Conway et al. [6, p. 132].

(iii) This follows from the proof of (i) and [30, Lemma 5.7(ii)]. □

2.8. Notation. We use the notation S, T, S_1, S_2, S_3 as in (2.6) and H as in (2.7). We fix a 3-block B of H corresponding to A via the Brauer correspondence. Namely, $B^G = A$. We denote $J(kH)$ by $\tilde{J}$.

2.9. Lemma. (i) *We can write* $\mathrm{Irr}(B) = \{\tilde{\chi}_5, \tilde{\chi}_6, \tilde{\chi}_{13}, \tilde{\chi}_{14}, \tilde{\chi}_{18}, \tilde{\chi}_{25}\}$ *where $\tilde{\chi}_i$ means an irreducible ordinary character of H such that*

Centr.	25 920	576	2880	64	324	3240	81	81	288	1440
class	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$3C$	$3D$	$4A$	$4B$
$\tilde{\chi}_5$	9	1	9	1	.	9	.	.	1	9
$\tilde{\chi}_6$	9	1	9	1	.	9	.	.	1	9
$\tilde{\chi}_{13}$	9	1	9	1	.	9	.	.	1	−9
$\tilde{\chi}_{14}$	9	1	9	1	.	9	.	.	1	−9
$\tilde{\chi}_{18}$	18	2	−18	−2	.	18	.	.	2	.
$\tilde{\chi}_{25}$	72	8	.	.	.	−9	.	.	8	.

(cont.)

Centr.	32	32	16	8	8	180	36	72	16	16
class	$4C$	$4D$	$4E$	$4F$	$4G$	$5A$	$6A$	$6B$	$8A$	$8B$
$\tilde{\chi}_5$	1	1	1	1	1	−1	.	1	−1	−1
$\tilde{\chi}_6$	1	1	1	−1	−1	−1	.	1	1	1
$\tilde{\chi}_{13}$	1	−1	−1	1	−1	−1	.	1	−1	−1
$\tilde{\chi}_{14}$	1	−1	−1	−1	1	−1	.	1	1	1
$\tilde{\chi}_{18}$	−2	.	.	.	.	−2	.	2	.	.
$\tilde{\chi}_{25}$	.	.	.	.	.	−8	.	−1	.	.

(cont.)

Centr.	16	16	20	36	36	36	45	45	20	20
class	$8C$	$8D$	$10A$	$12A$	$12B$	$12C$	$15A$	$15B$	$20A$	$20B$
$\tilde{\chi}_5$	−1	−1	−1	1	.	.	−1	−1	−1	−1
$\tilde{\chi}_6$	1	1	−1	1	.	.	−1	−1	−1	−1
$\tilde{\chi}_{13}$	1	1	−1	1	.	.	−1	−1	1	1
$\tilde{\chi}_{14}$	−1	−1	−1	1	.	.	−1	−1	1	1
$\tilde{\chi}_{18}$	.	.	2	2	.	.	−2	−2	.	.
$\tilde{\chi}_{25}$	.	.	.	−1	.	.	1	1	.	.

(ii) $\sigma \downarrow_H \in \mathrm{Aut}(H)$ *acts on the set* $\mathrm{Irr}(B)$ *such that* $\tilde{\chi}_{13}^{\sigma} = \tilde{\chi}_{14}$ *and* $\tilde{\chi}^{\sigma} = \tilde{\chi}$ *for all* $\tilde{\chi} \in \mathrm{Irr}(B) - \{\tilde{\chi}_{13}, \tilde{\chi}_{14}\}$.

(iii) *The relations of irreducible characters in* $\mathrm{Irr}(A)$ *and* $\mathrm{Irr}(B)$ *with respect to inductions and restrictions are the following*:

	$\tilde{\chi}_5$	$\tilde{\chi}_6$	$\tilde{\chi}_{13}$	$\tilde{\chi}_{14}$	$\tilde{\chi}_{18}$	$\tilde{\chi}_{25}$
χ_{10944}	6	6	5	5	6	33
χ_{52668}	22	20	20	20	34	148
χ_{58311_1}	25	21	21	21	38	165
χ_{58311_2}	21	22	21	22	38	163
χ_{58311_3}	21	22	22	21	38	163
χ_{58653}	18	20	20	20	42	161

(iv) *Let* $3A$ *be a unique conjugacy class of* G *with elements of order* 3, *and let* $\widetilde{3B}$ *be a conjugacy class of* H *in* (i). *Then*, $C_G(3A) = C_H(\widetilde{3B}) = D \times A_6$.

Proof. Conditions (i) and (iii) are obtained by GAP [37]. Condition (ii) is easy by Conway et al. [6, p. 133], (i), and (2.4)(iii). Condition (iv) is easy by (i) and (2.7)(i). □

2.10. Notation. We use the notation $\tilde{\chi}_i$'s for each i as in (2.9). Moreover, let $H' = D : Q_8$, $B' = B_0(kH') = kH'$, and let $\mathrm{IBr}(kH') = \{1_0 = k, 1_1, 1_2, 1_3, 2\}$ where each 1_i has k-dimension one and 2 has k-dimension two, see Okuyama [28, Section 4, Case 3].

2.11. Lemma. *We get the following*: (i) $\hat{B} \cong \mathrm{Mat}_9(\hat{B}')$ *as* $\mathcal{O}$-*algebras, where* $\hat{B}$ *and* $\hat{B}'$ *are the block algebras of* $\mathcal{O}H$ *and* $\mathcal{O}H'$ *corresponding to* B *and* B', *respectively. Furthermore*, $\hat{B}'$ *is isomorphic to a source algebra of* $\hat{B}$ *as interior D-algebras.*

(ii) *We can write* $\mathrm{IBr}(B) = \{\tilde{\varphi}_5, \tilde{\varphi}_6, \tilde{\varphi}_{13}, \tilde{\varphi}_{14}, \tilde{\varphi}_{18}\}$ *such that the* 3-*decomposition matrix of B has the following form*:

	$\tilde{\varphi}_5$	$\tilde{\varphi}_6$	$\tilde{\varphi}_{13}$	$\tilde{\varphi}_{14}$	$\tilde{\varphi}_{18}$
$\tilde{\chi}_5$	1	.	.	.	.
$\tilde{\chi}_6$	.	1	.	.	.
$\tilde{\chi}_{13}$	.	.	1	.	.
$\tilde{\chi}_{14}$	.	.	.	1	.
$\tilde{\chi}_{18}$	.	.	.	.	1
$\tilde{\chi}_{25}$	1	1	1	1	2

so that $\dim_k \tilde{\varphi}_i = 9$ *for* $i = 5, 6, 13, 14$ *and* $\dim_k \tilde{\varphi}_{18} = 18$.

(iii) *The Cartan matrix of B has the following form*:

	$P(\tilde{\varphi}_5)$	$P(\tilde{\varphi}_6)$	$P(\tilde{\varphi}_{13})$	$P(\tilde{\varphi}_{14})$	$P(\tilde{\varphi}_{18})$
$\tilde{\varphi}_5$	2	1	1	1	2
$\tilde{\varphi}_6$	1	2	1	1	2
$\tilde{\varphi}_{13}$	1	1	2	1	2
$\tilde{\varphi}_{14}$	1	1	1	2	2
$\tilde{\varphi}_{18}$	2	2	2	2	5

(iv) *The Loewy and socle series for the projective indecomposable kH-modules in B are the following*:

$$P(\tilde{\varphi}_i) = \begin{matrix} \tilde{\varphi}_i \\ \tilde{\varphi}_{18} \\ \tilde{\varphi}_j \ \tilde{\varphi}_{k'} \ \tilde{\varphi}_\ell \\ \tilde{\varphi}_{18} \\ \tilde{\varphi}_i \end{matrix} \quad \text{for } \{i, j, k', \ell\} = \{5, 6, 13, 14\}, \quad \textit{and}$$

$$P(\tilde{\varphi}_{18}) = \begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_5 \ \tilde{\varphi}_6 \ \tilde{\varphi}_{13} \ \tilde{\varphi}_{14} \\ \tilde{\varphi}_{18} \ \tilde{\varphi}_{18} \ \tilde{\varphi}_{18} \\ \tilde{\varphi}_5 \ \tilde{\varphi}_6 \ \tilde{\varphi}_{13} \ \tilde{\varphi}_{14} \\ \tilde{\varphi}_{18} \end{matrix} .$$

(v) *There are exactly five trivial source kH-modules in B such that their vertices are D. They are, actually, all simple, namely,* $\tilde{\varphi}_5, \tilde{\varphi}_6, \tilde{\varphi}_{13}, \tilde{\varphi}_{14}$ *and* $\tilde{\varphi}_{18}$.

(vi) $\sigma\downarrow_H \in \mathrm{Aut}(H) - \mathrm{Inn}(H)$ *acts on the set* $\mathrm{IBr}(B)$ *such that* $\tilde{\varphi}_{13}^{\sigma} = \tilde{\varphi}_{14}$ *and* $\tilde{\varphi}^{\sigma} = \tilde{\varphi}$ *for all* $\tilde{\varphi} \in \mathrm{IBr}(B) - \{\tilde{\varphi}_{13}, \tilde{\varphi}_{14}\}$.

Proof. Condition (i) follows from (2.7)(i), results of Külshammer [20, A.Theorem], Puig [31, Proposition 14.6] and [13, V, 25.3 Satz(b)]. Conditions (ii)–(v) are easy since the defect group D of B is normal in H (see [22, III, Lemma 10.3]). □

2.12. Notation. We use the notation $\tilde{\varphi}_i$'s as in (2.11).

2.13. Lemma. *By Conway et al.* [6, p. 132] (or [40,41]), *G has maximal subgroups L and M such that* $L = L_3(7)\cdot 2$ *and* $M = 4\cdot L_3(4)\cdot 2$. *We then get the following*:

(i) $1_L \uparrow^G \cdot 1_{\hat{A}} = \chi_{10944} + \chi_{52668}$.

(ii) $1_M \uparrow^G \cdot 1_{\hat{A}} = 2\times\chi_{10944} + 2\times\chi_{52668} + 2\times\chi_{58311_1} + \chi_{58311_2} + \chi_{58311_3}$.

(iii) $(-1)_M \uparrow^G \cdot 1_{\hat{A}} = \chi_{10944} + \chi_{58311_1} + \chi_{58653}$, *where* $(-1)_M$ *is a non-trivial linear ordinary character of M afforded by the extension* $M \triangleright 4\cdot L_3(4)$.

Proof. We get the assertions by using GAP [37]. □

2.14. Lemma. *Let* $Q = \langle\widetilde{3B}\rangle \cong C_3$ *with* $Q \lneqq D$ *where* $\widetilde{3B}$ *is the same as in* (2.9)(i).

(i) *There are exactly two trivial source kH-modules in B with vertex Q. They are, actually,*

$$\begin{matrix} \tilde{\varphi}_5\ \tilde{\varphi}_6\ \tilde{\varphi}_{13}\ \tilde{\varphi}_{14} \\ \tilde{\varphi}_{18}\ \tilde{\varphi}_{18} \\ \tilde{\varphi}_5\ \tilde{\varphi}_6\ \tilde{\varphi}_{13}\ \tilde{\varphi}_{14} \end{matrix} \leftrightarrow \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_{13} + \tilde{\chi}_{14} + \tilde{\chi}_{25},$$

$$\begin{matrix} \tilde{\varphi}_{18}\ \tilde{\varphi}_{18} \\ \tilde{\varphi}_5\ \tilde{\varphi}_6\ \tilde{\varphi}_{13}\ \tilde{\varphi}_{14} \\ \tilde{\varphi}_{18}\ \tilde{\varphi}_{18} \end{matrix} \leftrightarrow 2\times\tilde{\chi}_{18} + \tilde{\chi}_{25}.$$

(ii) *Let X and X′ be the modules in* (i). *Then*, $P_Q(\tilde{\varphi}_i) = X$ *for* $i = 5, 6, 13, 14$, *and* $P_Q(\tilde{\varphi}_{18}) = X'$.

Proof. (i) By (2.7)(i), H has a subgroup K such that $K = (Q:2)\times A_6 = \Sigma_3 \times A_6$, and hence $|H:K| = 12$. Let ψ and ψ' be ordinary characters of Σ_3 of degree 1 such that ψ is the trivial character, and let $\chi \in \mathrm{Irr}(A_6)$ with $\chi(1) = 9$. Then, $\psi\otimes\chi$, $\psi'\otimes\chi \in \mathrm{Irr}(K)$. We know that $\psi\uparrow^H = \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_{13} + \tilde{\chi}_{14} + \tilde{\chi}_{25}$ and $\psi'\uparrow^H = 2\times\tilde{\chi}_{18} + \tilde{\chi}_{25}$ by elementary calculations or just by GAP [37]. There are kK-modules $1\otimes 9$ and $1'\otimes 9$ afforded by $\psi\otimes\chi$ and $\psi'\otimes\chi$, respectively. Both kK-modules are trivial source modules with vertex Q. Let $X = (1\otimes 9)\uparrow^H$ and $X' = (1'\otimes 9)\uparrow^H$. Thus, it follows from the above decompositions of $\psi\uparrow^H$ and $\psi'\uparrow^H$, Frobenius reciprocity, (1.1)(ii), (2.11)(ii) and (2.11)(iv) that

$$X = \begin{matrix} \tilde{\varphi}_5\ \tilde{\varphi}_6\ \tilde{\varphi}_{13}\ \tilde{\varphi}_{14} \\ \tilde{\varphi}_{18}\ \tilde{\varphi}_{18} \\ \tilde{\varphi}_5\ \tilde{\varphi}_6\ \tilde{\varphi}_{13}\ \tilde{\varphi}_{14} \end{matrix} \quad \text{(Loewy and socle series).}$$

Since the two composition factors $\tilde{\varphi}_{18}$ of $L_2(X)$ come from only $\tilde{\chi}_{25}$ by (2.11)(ii), X is indecomposable by (1.1)(i). Similarly, we know that

$$X' = \begin{matrix} & \tilde{\varphi}_{18} & \tilde{\varphi}_{18} & \\ \tilde{\varphi}_5 & \tilde{\varphi}_6 & \tilde{\varphi}_{13} & \tilde{\varphi}_{14} \\ & \tilde{\varphi}_{18} & \tilde{\varphi}_{18} & \end{matrix} \quad \text{(Loewy and socle series).}$$

Then, since the composition factors $\tilde{\varphi}_5$, $\tilde{\varphi}_6$, $\tilde{\varphi}_{13}$, $\tilde{\varphi}_{14}$ of $L_2(X')$ come from only $\tilde{\chi}_{25}$ by (2.11)(ii), X' is also indecomposable by (1.1)(i). By (2.9)(iv) and (2.7) (or just by GAP [37]), $N_H(Q)/Q = [(D/Q) \times A_6] \cdot 2$. Let b be a block of $N_H(Q)$ with $B = b^H$. Then, $\ell(b) = 2$, and hence X and X' are all trivial source kH-modules in B with vertex Q by Nagao and Tsushima [26, Chapter 4, Problem 10] and (1.7).

(ii) By (2.11)(v) and the definition of Q-projective covers in [17, p. 29], $P_Q(\tilde{\varphi}_i)$ is a trivial source module in B with vertex Q for each i. Hence, (i) implies the assertion. □

2.15. Lemma. *With the notation in* (2.10) *we get the following*: (i) *There does not exist a* kH'*-module* $U(2, 1_i, 2)$ *for any* $i = 1, 2, 3$.

(ii) *For each* $i = 1, 2, 3$ *there exists a* kH'*-module whose Loewy and socle series are*

$$\begin{matrix} & 2 & \\ k & & 1_i \\ & 2 & \end{matrix}$$

uniquely up to isomorphism.

Proof. (i) Suppose that there is a kH'-module $U(2, 1_1, 2)$. Then, by using an automorphism of Q_8 of order 3, there are kH'-modules $U(2, 1_i, 2)$ for $i = 2, 3$, too. Let $X_i = U(2, 1_i, 2)$ and we may assume that $X_i \subseteq P(2)$ for $i = 1, 2, 3$. Then,

$$X_1 + X_2 + X_3 = \begin{matrix} 2 & 2 & 2 \\ 1_1 & 1_2 & 1_3 \\ & 2 & \end{matrix}.$$

Thus, we have a contradiction by looking at the structure of $P(2)$ and $P(2)/(X_1 + X_2 + X_3)$, see (2.11)(iv) and note that B and B' are Morita equivalent by (2.11)(i).

(ii) Assume that there are kH'-modules U and V such that U and V have the same Loewy and socle series

$$\begin{matrix} & 2 & \\ k & & 1_1 \\ & 2 & \end{matrix}$$

and $U \not\cong V$. We may consider U, $V \subseteq P(2)$. Let $W = U + V$. Clearly, $U \neq V$. Hence,

$$W = \begin{matrix} 2 & & 2 \\ k & & 1_1 \\ & 2 & \end{matrix}.$$

Now, since $\dim_k[\mathrm{Ext}^1_{kH'}(2, k)] = 1$ by (2.11)(iv) and (2.11)(i), W has a submodule W' with $W' = U(2, 1_1, 2)$, contradicting (i). □

2.16. Lemma. *We use the notation in* (2.10). *Let* $M = (1_1)_{\Delta H'} \uparrow^{H' \times H'}$, *so that* M *may be considered as* (kH', kH')*-bimodule. Then*, M *induces an auto-equivalence of* mod-kH', *namely*, *a Morita equivalence between* mod-kH' *and itself. In other words*, *a functor* $F : X \mapsto X \otimes_{kH'} M$ *for* $X \in$ mod-kH' *realizes an auto-equivalence*, *and it holds that* $F(1_0) = 1_1, F(1_1) = 1_0, F(1_2) = 1_3, F(1_3) = 1_2$ *and* $F(2) = 2$. *Moreover*, *this equivalence is a Puig* (*splendid Morita*) *equivalence.*

Proof. It follows from (1.10) that, for $X \in$ mod-kH', $F(X) = X \otimes_{kH'} M = (X \otimes_k 1_1)_{kH'}$. Thus, we know $F(S')$ for each simple S' in mod-kH' as desired, and moreover $F^2 = \mathrm{Id}$ since $1_1 \otimes_k 1_1 = k_{H'}$. Since 1_1 is a ΔD-projective 3-permutation $k[\Delta H']$-module, M realizes a Puig equivalence. □

3. Projective indecomposable modules in A

In this section we investigate Loewy and socle structure of several projective indecomposable kG-modules and the dimensions of $\mathrm{Ext}^1_{kG}(S', T')$ for simples S' and T' in A. The information helps so much to get our main result (0.2)(i). We keep all the notations used in Section 2, namely in (2.1), (2.3), (2.5), (2.8), (2.10) and (2.12).

3.1. Lemma. *For any simple kG-module* S' *in* A, $\mathrm{Ext}^1_{kG}(S', S') = 0$.

Proof. Since $k(A) - \ell(A) = 1$ by (2.4)(i) and (2.6)(i), the assertion follows from a result of Brandt [3, Theorem B]. □

3.2. Lemma. *Let* L *be as in* (2.13). *Then*, $k_L \uparrow^G \cdot 1_A = U(S, T, S) \leftrightarrow \chi_{10944} + \chi_{52668}$ *and this is a trivial source kG-module.*

Proof. Let $U = k_L \uparrow^G \cdot 1_A$. By (2.13)(i) and (1.1)(i), $\chi_{\hat{U}} = \chi_{10944} + \chi_{52668}$. If $T | \mathrm{Soc}(U)$, then the self-dualities imply $T | U$ since $c_U(T) = 1$, so that T is a trivial source module, contradicting (1.1)(i) and (2.6)(iv). Hence, $T \nmid \mathrm{Soc}(U)$, so that $T \nmid L_1(U)$ again by the self-dualities. Therefore U has Loewy and socle series as desired. □

3.3. Lemma. *For any simple kG-module* S' *in* A, *the heart* $\mathscr{H}(P(S')) = [P(S') \cdot J]/S'$ *of* $P(S')$ *is indecomposable as kG-module.*

Proof. Follows from [2, Theorem 4.4.4], (2.6)(v) and (1.9). □

3.4. Lemma. $\mathrm{Ext}^1_{kG}(S_i, S_j) = 0$ *for all* $i, j \in \{0, 1, 2, 3\}$, *possibly* $i = j$, *where* $S_0 = S$.

Proof. Note, first, that S_0, S_1, S_2, S_3 are all self-dual by (2.6)(ii). Since $c(S_i, S_j) = 1$ for $i \neq j$ by (2.6)(v), we get the assertion from (3.3), (3.1) and (2.6)(ii). □

3.5. Lemma. *There are kG-modules* $U(S, T, S_i)$ *for* $i = 1, 2, 3$.

Proof. Easy by (1.4), (2.6)(iv) and (3.4). □

3.6. Lemma. $\dim_k[\mathrm{Ext}^1_{kG}(S_i, T)] = 1$ *for* $i = 1, 2, 3$.

Proof. Easy by (3.5), (2.6)(ii), (3.3) and (2.6)(v). □

3.7. Lemma. *The Loewy and socle series of projective indecomposable modules* $P(S_i)$ *for* $i = 1, 2, 3$ *are the following*:

$$P(S_i) = \begin{matrix} S_i \\ T \\ SS_jS_{k'} \\ T \\ S_i \end{matrix} \quad \textit{for } \{i, j, k'\} = \{1, 2, 3\}.$$

Proof. By (2.6)(i), $\mathrm{IBr}(A) = \{S, T, S_1, S_2, S_3\}$. Let $P_1 = P(S_1)$. It follows from (3.1), (3.4), (3.5) and (3.6) that $L_2(P_1) = T$ and $S | L_3(P_1)$. Thus, by (2.6)(iv) and (1.4), P_1 has a factor module W such that $W = T + S_1 + S_2 + S_3$, as composition factors. Then, (3.4) implies that $j(W) = 3$ and $L_i(W)$ is S_1, T, $S_2 \oplus S_3$ for $i = 1, 2, 3$, respectively. By (3.1), we get $T \nmid L_3(P_1)$. So, P_1 has Loewy and socle series as desired by (2.6)(ii). Similarly for $P(S_2)$ and $P(S_3)$. □

3.8. Remark. After proving (0.3), we will be able to know the Loewy and socle series of $P(S)$ and $P(T)$ which we do not prove above. We will not need this to prove one of our main results (0.2)(i), see (4.15).

4. Green correspondents of simples in *A* and *B*, and a proof of the main result

In this section we calculate structure of Green correspondents of simple kG-modules in A.

4.1. Lemma. *Let* M *be as in* (2.13), *and let* $Y = (-1)_M \uparrow^G \cdot 1_A$ *where* $(-1)_M$ *is a non-trivial one-dimensional* kM*-module given by the extension* $M \triangleright 4 \cdot L_3(4)$. *Then it follows* $Y = S \oplus P(S_1)$. *In particular*, S *is a trivial source* kG*-module affording* χ_{10944}.

Proof. By (2.13)(iii), (1.1)(i) and (2.6)(iv), it holds

$$\chi_{\hat{Y}} = \chi_{10944} + \chi_{58311_1} + \chi_{58653} \tag{1}$$

and

$$Y = S + (S + T + S_1) + (T + S_1 + S_2 + S_3), \quad \text{as composition factors.} \tag{2}$$

Then, (1) and a theorem of Scott (1.1)(ii) show that

$$[Y,\ Y]^G = 3. \tag{3}$$

It follows from (2.4)(ii) and (1.2)(ii) that only the following three cases can occur. Namely,

(a) $Y = Y_1 \oplus Y_2$ for indecomposable kG-modules Y_1 and Y_2 such that $\chi_{\hat{Y}_1} = \chi_{10944} + \chi_{58311_1}$ and $\chi_{\hat{Y}_2} = \chi_{58653}$.

(b) Y is indecomposable.

(c) $Y = Y_1 \oplus Y_2$ for indecomposable kG-modules Y_1 and Y_2 such that $\chi_{\hat{Y}_1} = \chi_{10944}$ and $\chi_{\hat{Y}_2} = \chi_{58311_1} + \chi_{58653}$.

Case (a): Then $\chi_{\hat{Y}_1}(3A) = 0$ by (2.4)(ii), which means from (1.2)(iii) that Y_1 is a projective kG-module, contradicting (2.6)(iv).

Case (b): From (2.6)(iv) and (1.1)(i), S_i is not a trivial source module for any $i = 1,2,3$. Since $c_Y(S_2) = c_Y(S_3) = 1$, (2.6)(ii) implies that

$$S_i \nmid L_1(Y) \quad \text{and} \quad S_i \nmid \mathrm{Soc}(Y) \quad \text{for } i = 2,3. \tag{4}$$

By (2.6)(iv), χ_{10944} affords S, so that (1.3) implies

$$S|L_1(Y) \quad \text{and} \quad S|\mathrm{Soc}(Y). \tag{5}$$

Again from (2.6)(iv), Y is not projective.

Suppose $\mathrm{Soc}(Y) = S$. Then, $Y \hookrightarrow P(S)$. Thus, it follows from (2) and (2.6)(v) that $2 = c_Y(S_1) \leqslant c(S,S_1) = 1$, a contradiction. Hence, $\mathrm{Soc}(Y) \neq S$.

Next, assume that there are three simple kG-modules L_1, L_2 and L_3 such that $(L_1 \oplus L_2 \oplus L_3)|\mathrm{Soc}(Y)$. Then, since $(L_1 \oplus L_2 \oplus L_3) \subsetneq Y$, we have $[Y,Y]^G \geqslant 4$ by (2.6)(ii), contradicting (3). Hence, $\mathrm{Soc}(Y)$ contains at most two direct summands of simple modules.

Therefore, by (5), we have $\mathrm{Soc}(Y) = S \oplus V$ for a simple kG-module V. If $V \cong S$, then $[Y,Y]^G \geqslant 5$ by (2.6)(ii), contradicting (3). Hence, by (2) and (4), $\mathrm{Soc}(Y) = S \oplus S_1$ or $S \oplus T$.

Assume $\mathrm{Soc}(Y) = S \oplus T$. By (2.6)(ii) and the self-duality of Y, $L_1(Y) = S \oplus T$. First of all, we claim that $\mathrm{Soc}(Y) \subseteq YJ$. Suppose $\mathrm{Soc}(Y) \nsubseteq YJ$. Let $Y' = YJ + \mathrm{Soc}(Y)$. If $Y' = Y$, then Nakayama's lemma implies that Y is semi-simple, a contradiction. Hence, $Y' \subsetneq Y$. If $Y/Y' = T$ and $Y'/YJ = S$, then we know $YJ/[\mathrm{Soc}(Y) \cap YJ] = S \oplus S_1 \oplus S_1 \oplus S_2 \oplus S_3$ from (2) and (3.4), which means that $\dim_k[\mathrm{Ext}^1_{kG}(S \oplus T, S_1)] \geqslant 2$, contradicting (3.7). Hence, $Y/Y' = S$ and $Y'/YJ = T$. Thus, $\mathrm{Soc}(YJ) = \mathrm{Soc}(Y) \cap YJ = S$, and therefore $YJ \hookrightarrow P(S)$. On the other hand, (2) shows that $c_{YJ}(S_1) = 2$, which is a contradiction since $c(S,S_1) = 1$ by (2.6)(v). Now, we know that $\mathrm{Soc}(Y) \subseteq YJ$. Thus, $YJ/\mathrm{Soc}(Y) = S_1 \oplus S_1 \oplus S_2 \oplus S_3$ by (2) and (3.4). Hence, $\dim_k[\mathrm{Ext}^1_{kG}(S \oplus T, S_1)] \geqslant 2$. This is a contradiction by (3.7) just as above.

Therefore, $\mathrm{Soc}(Y) = S \oplus S_1$.Then, $L_1(Y) = S \oplus S_1$ as above. Then, (2) and (3.4) imply that $L_2(Y) = T \oplus T$ or T. If $L_2(Y) = T \oplus T$, then we know that $j(Y) = 3$ and $L_3(Y) = S \oplus S_1 \oplus S_2 \oplus S_3$ by (3.4), and hence $(S_2 \oplus S_3)|\mathrm{Soc}(Y)$, a contradiction. Thus, $L_2(Y) = T$. Clearly, there is an epimorphism $f : P(S) \oplus P(S_1) \twoheadrightarrow Y$. It follows from (3.7), Landrock's lemmas (1.5), (2.6)(ii) and (2.6)(v) that $c(S,S_i) = c(S_1,S_i) = 1$, $S_i|L_3(P(S))$ and $S_i|L_3(P(S_1))$ for $i = 2,3$. This means that $(S_2 \oplus S_3)|L_3(Y)$ by (2). Hence, $(S_2 \oplus S_3)|[\mathrm{Soc}_3(Y)/\mathrm{Soc}_2(Y)]$ by the self-duality of Y and (2.6)(ii), so that $j(Y) \geqslant 5$ since $c_Y(S_i) = 1$ for $i = 2,3$ by (2). Now, as above, $S_1|L_3(P(S))$ from (3.7) and (1.5). Thus, by (3.7) again, we know that $S_1|L_3(Y)$ or $S_1|L_5(Y)$. But, if $S_1|L_5(Y)$, then $P(S_1)|Y$ by using the epimorphism f, so that Y is decomposable by (2), a contradiction.

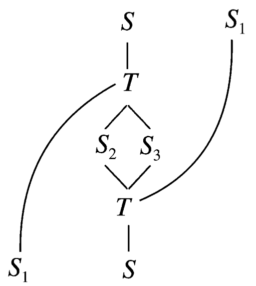
Therefore, $S_1 | L_3(Y)$. Since $T \nmid \mathrm{Soc}(Y)$, the Loewy series of YJ^2 is

$$\begin{array}{c} S_1\ S_2\ S_3\ T \\ S \end{array} \quad \text{or} \quad \begin{array}{c} S_1\ S_2\ S_3 \\ T \\ S \end{array}$$

If the first case occurs, then (3.4) implies that $(S_2 \oplus S_3)|\mathrm{Soc}(Y)$, a contradiction. Thus, only the second case happens. Therefore, the self-dualities show that Y has Loewy and socle series, respectively,

$$\begin{array}{c} S\ S_1 \\ T \\ S_1\ S_2\ S_3 \\ T \\ S \end{array} \quad \text{and} \quad \begin{array}{c} S \\ T \\ S_1\ S_2\ S_3 \\ T \\ S\ S_1 \end{array}$$

Let $\mathrm{Soc}(Y) = U \oplus U_1$ such that $U \cong S$ and $U_1 \cong S_1$, and let Z be a submodule of Y such that $YJ \subsetneq Z$ and $Z/YJ = S$. If $U_1 \nsubseteq YJ$, then $U_1 \nsubseteq Z$, so that $U_1 \cap Z = 0$, which implies that $Y = U_1 \oplus Z$ by (2), a contradiction since Y is indecomposable. Hence $U_1 \subseteq YJ$. Therefore, since there does not exist a module $U(S_1, T, S_1)$ by (3.7), it follows by (3.4) that Y has structure of form

$Y =$ 

(6)

Now, since we have known that $[L_i(P(S)), S_2 \oplus S_3]^G = 1$ if $i = 3$ and 0 if $i \neq 3$, the self-dualities show that $[S_2 \oplus S_3, \mathrm{Soc}_i(P(S))/\mathrm{Soc}_{i-1}(P(S))] = 1$ if $i = 3$ and 0 if $i \neq 3$. Thus, we know $j(P(S)) = 5$ by (6) since $c(S, S_2) = c(S, S_3) = 1$ by (2.6)(v). By (6), Y has a submodule Z such that $Y/Z \cong S_1$ and $S_1 \hookrightarrow Z$. Let $W = Z/S_1$. Then, it follows from (6) that $L_1(W) \cong \mathrm{Soc}(W) \cong S$ and $j(W) = 5$. Hence, $W = P(S)$ since $j(P(S)) = 5$ from the above. This is a contradiction by (2.6)(v).

Therefore, only Case (c) occurs. Hence, (2.6)(iv) implies that $Y_1 \cong S$. Now, $\chi_{\hat{Y}_2}(3A) = 0$ from (2.4)(ii). Hence, as in the proof of Case (a), Y_2 is a projective indecomposable kG-module. Thus, again by (2.6)(iv), $Y_2 = P(S_1)$. This completes the proof of (4.1). □

4.2. Notation. In the rest of this paper except Section 5, let $f : G \to H$ and $g : H \to G$ be the Green correspondences with respect to (G, D, H).

4.3. Lemma. (i) *For any non-projective indecomposable kG-module U in A*, $U \downarrow_H \cdot 1_B = fU \oplus (\mathrm{proj})$.

(ii) *For any non-projective indecomposable kH-module V in B*, $V\uparrow^G\cdot 1_A = gV \oplus$ (proj).

Proof. Easy by (2.7)(iii) and Nagao and Tsushimo [26, Chapter 4, Theorem 4.3(i)]. □

4.4. Lemma. *It holds the following*:

(i) $S\downarrow_H\cdot 1_B = \tilde{\varphi}_6 \oplus (6\times P(\tilde{\varphi}_5)\oplus 5\times P(\tilde{\varphi}_6)\oplus 5\times P(\tilde{\varphi}_{13})\oplus 5\times P(\tilde{\varphi}_{14})\oplus 6\times P(\tilde{\varphi}_{18}))$. *In particular*, $fS=\tilde{\varphi}_6$ *and* $g\tilde{\varphi}_6 = S$.

(ii) $\tilde{\varphi}_6\uparrow^G\cdot 1_A = S\oplus(5\times P(S)\oplus 15\times P(T)\oplus P(S_1)\oplus 2\times P(S_2)\oplus 2\times P(S_3))$.

Proof. By (1.7), (4.1) and [27, Lemma 2.2] by Okuyama, fS is a simple kH-module in B.

Assume that $fS=\tilde{\varphi}_{18}$. Then, by (4.3)(ii), $\tilde{\varphi}\uparrow^G\cdot 1_A = S\oplus$ (proj). (2.9)(iii) implies that $[(\tilde{\chi}_{18}\uparrow^G)\cdot 1_{\hat{A}}](1)\equiv 18 \pmod{81}$. This is a contradiction by (2.6)(iv) and (2.11)(ii).

Therefore, $fS\in\{\tilde{\varphi}_5,\tilde{\varphi}_6,\tilde{\varphi}_{13},\tilde{\varphi}_{14}\}$ by (2.11)(ii). We then obtain by (2.9)(iii) and (2.11)(ii) that $[P(\tilde{\varphi}_{18})|S\downarrow_H] = 6$.

Assume, first, that $fS=\tilde{\varphi}_5$. Then, $g\tilde{\varphi}_5 = S$. Note that $\tilde{\varphi}_5\uparrow^G\cdot 1_A = S\oplus P$ for a projective kG-module P by (4.3)(ii). Thus, by (2.9)(iii), we know

$$\begin{aligned}\Phi_P &= 5\times\chi_{10944}+22\times\chi_{52668}+25\times\chi_{58311_1}+21\times\chi_{58311_2}\\ &\quad +21\times\chi_{58311_3}+18\times\chi_{58653}.\end{aligned}$$

Then, it follows by (2.6)(iv) that $[P(S)|\tilde{\varphi}_5\uparrow^G] = 5$, so that $[P(T)|\tilde{\varphi}_5\uparrow^G] = 17$. However,

$$(\Phi_P - 5\times\Phi_{P(S)} - 17\times\Phi_{P(T)}, \chi_{58311_2})_G = 21-5-17<0$$

from (2.6)(iv). This is a contradiction. Hence, $fS\neq\tilde{\varphi}_5$.

Suppose, next, that $fS=\tilde{\varphi}_{13}$. This is a contradiction since σ fixes S but does not fix $\tilde{\varphi}_{13}$ from (2.6)(iii) and (2.11)(vi). Hence, $fS\neq\tilde{\varphi}_{13}$.

Similarly, we get $fS\neq\tilde{\varphi}_{14}$. Therefore, $fS=\tilde{\varphi}_6$. The rest is obtained similarly as above. □

4.5. Lemma. (i) *There are exactly five trivial source kG-modules in A with vertex D, which are* $g\tilde{\varphi}_5$, $g\tilde{\varphi}_6$, $g\tilde{\varphi}_{13}$ $g\tilde{\varphi}_{14}$ *and* $g\tilde{\varphi}_{18}$.

(ii)

$$\dim_k g\tilde{\varphi}_i \equiv \begin{cases} 9 \pmod{81} & \textit{for } i=5,6,13,14,\\ 18\pmod{81} & \textit{for } i=18.\end{cases}$$

(iii)

$$[S, g\tilde{\varphi}_i]^G = [g\tilde{\varphi}_i, S]^G = \begin{cases} 1 & \textit{for } i=6,\\ 0 & \textit{for } i=5,13,14,18.\end{cases}$$

Proof. (i) This follows from [26, Chapter 4, Problem 10] and (2.11)(v).

(ii) Easy by (2.11)(ii) and (2.7)(iii) since the order of a Sylow 3-subgroup is 81.

(iii) By (1.6), we know $[S, \tilde{\varphi}_i \uparrow^G]^G = [S\downarrow_H, \tilde{\varphi}_i]^H$ and $[P(S)|\tilde{\varphi}_i \uparrow^G] = [P(\tilde{\varphi}_i)|S\downarrow_H]$ for $i = 5, 6, 13, 14, 18$. Hence, (4.4)(i)–(ii) implies that the values $[S, g\tilde{\varphi}_i]^G$ are as desired. □

4.6. Lemma. *Let M be as in* (2.13), *and let $X = k_M \uparrow^G \cdot 1_A$. Then,*

$$\begin{array}{ccccccc} & & & & T & & \\ X = & S & \oplus & & SSS_1 & & (\text{Loewy and socle series}) \oplus P(S). \\ & & & & T & & \\ & \updownarrow & & & \updownarrow & & \\ & \chi_{10944} & & & \chi_{52668} + \chi_{58311_1} & & \end{array}$$

Proof. It follows from (1.1)(i) and (2.13)(ii) that

$$\chi_{\hat{X}} = 2 \times \chi_{10944} + 2 \times \chi_{52668} + 2 \times \chi_{58311_1} + \chi_{58311_2} + \chi_{58311_3}. \tag{7}$$

Hence, by (1.1)(ii), (4.1) and (7), we know

$$[S, X]^G = [X, S]^G = 2. \tag{8}$$

Since $c_X(S) = 8 > 2$ by (7) and (2.6)(iv), (8) shows that

$$j(X) \geqslant 3. \tag{9}$$

(2.4)(ii) and (7) imply

$$\chi_{\hat{X}}(3A) = 18. \tag{10}$$

First, suppose that X is indecomposable as kG-module. Since $v_3(\dim_k X) = 2$, we get by (1.8) that D is a vertex of X. By (7), $\dim_k X \equiv 18 \pmod{81}$. Thus, (4.5)(i)–(ii) shows $X \cong g\tilde{\varphi}_{18}$, so that $[S, X]^G = 0$ by (4.5)(iii), contradicting (8).

Therefore, X is decomposable.

Next, assume that X is projective-free. Then, X is written as $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ for an integer $n \geqslant 2$ and indecomposable kG-modules $X_1, \ldots, X_n$. By (1.2)(iii), $\chi_{\hat{X}_i}(3A) > 0$ for all i. Hence, (2.4)(ii) and (7) imply that $\chi_{\hat{X}_i}(3A) = 9m_i$ for an integer $m_i \geqslant 1$ for each i. Thus, $n = 2$ by (10). Namely, $X = X_1 \oplus X_2$ and $\chi_{\hat{X}_i}(3A) = 9$ for $i = 1, 2$. Now, by (7) and (2.4)(ii), there are exactly two 18's, two 9's and four -9's appearing in the values $\chi_{\hat{X}}(3A)$. Therefore, only the following four cases occur:

	$\chi_{\hat{X}_1}(3A)$	$\chi_{\hat{X}_2}(3A)$
(a)	$18 + 9 - 9 - 9$	$18 + 9 - 9 - 9$
(b)	$18 - 9$	$18 + 9 + 9 - 9 - 9 - 9$
(c)	$9 + 9 - 9$	$18 + 18 - 9 - 9 - 9$
(d)	9	$18 + 18 + 9 - 9 - 9 - 9 - 9$

Case (a): By (7) and (2.4)(ii), $\chi_{\hat{X}_1} = \chi_{10944} + \chi_{52668} + \chi_{58311_i} + \chi_{58311_j}$ for some $i, j \in \{1, 2, 3\}$ (possibly, $i = j = 1$). Then, $v_3(\dim_k X_1) = 2$, so that D is a vertex of

X_1 from (1.8). Since $\dim_k X_1 \equiv 9 \pmod{81}$, (4.5)(i)–(ii) implies $X_1 \in \{g\tilde{\varphi}_5, g\tilde{\varphi}_6, g\tilde{\varphi}_{13}, g\tilde{\varphi}_{14}\}$. On the other hand, by (1.1)(ii) and (4.1), we know $[S, X_1]^G = (\chi_{10944}, \chi_{\hat{X}_1})_G = 1$. These imply $X_1 = g\tilde{\varphi}_6$, so that $X_1 = S$ from (4.4)(i). Hence, $\chi_{\hat{X}_1} = \chi_{\hat{S}} = \chi_{10944}$ by (4.1) and (2.6)(iv), a contradiction.

Case (b): As in Case (a), we get $\chi_{\hat{X}_2} = 2 \times \chi_{10944} + \chi_{52668} + \chi_{58311_i} + \chi_{58311_j} + \chi_{58311_{k'}}$ for $i, j, k' \in \{1, 2, 3\}$. Then, just as in Case (a), $X_2 \in \{g\tilde{\varphi}_5, g\tilde{\varphi}_6, g\tilde{\varphi}_{13}, g\tilde{\varphi}_{14}\}$. Hence, $[S, X_2]^G = (\chi_{10944}, \chi_{\hat{X}_2})_G = 2$ as in Case (a), contradicting (4.5)(iii).

Case (c): As in Case (a), $(\chi_{\hat{X}_1}, \chi_{10944})_G = 2$. Hence, we get a contradiction by (4.5)(iii) just as in Case (b).

Case (d): As in Case (a), $X_1 \cong X_2 \cong S$. Hence $\chi_{\hat{X}} = 2 \times \chi_{10944}$, contradicting (7).

Thus, X contains a non-zero projective direct summand. So, $= P(S') \oplus V$ for a simple kG-module S' in A and a kG-module $V \neq 0$. Since $(\chi_{\hat{X}}, \chi_{58653})_G = 0$ by (7), (2.6)(iv) implies that $S' = S$, so that $\chi_{\hat{V}} = \chi_{10944} + \chi_{52668} + \chi_{58311_1}$ by (7) and (2.6)(iv). Namely, $X = P(S) \oplus V$.

Assume V is indecomposable. Since $v_3(\dim_k V) = 2$, D is a vertex of V from (1.8). Clearly, $\dim_k V = 18 \pmod{81}$. Hence, as the above, $V = g\tilde{\varphi}_{18}$ by (4.5)(i)–(ii). Hence, $[S, V]^G = 0$ by (4.5)(iii). On the other hand, as the above, $[S, V]^G = (\chi_{10944}, \chi_{\hat{V}})_G = 1$, a contradiction.

Hence, V is decomposable. So, $V = V_1 \oplus V_2$ for kG-modules V_1 and V_2 such that $V_i \neq 0$ for $i = 1, 2$. It follows from (1.2)(iii) and (2.4)(ii) that we may assume $\chi_{\hat{V}_1} = \chi_{10944}$ and $\chi_{\hat{V}_2} = \chi_{52668} + \chi_{58311_1}$. Hence, (1.1)(i) and (2.6)(iv) say $V_1 = S$. Therefore, $X = P(S) \oplus S \oplus V_2$. Then, again by (1.2)(iii) and (2.4)(ii), V_2 is indecomposable. Clearly, (2.6)(iv) implies that $V_2 = (S+T) + (S+T+S_1)$, as composition factors. By (2.6)(iv) and (1.1)(i), S_1 is not a trivial source kG-module. Hence, $[S_1, V_2]^G = [V_2, S_1]^G = 0$ by the self-dualities. Thus, $j(V_2) \geqslant 3$. Moreover, by (4.1), (1.1)(ii) and (2.6)(iv), and also by the self-dualities, we know that $[S, V_2]^G = [V_2, S]^G = (\chi_{10944}, \chi_{\hat{V}_2})_G = 0$. Hence $L_1(V_2) \cong T \cong \mathrm{Soc}(V_2)$. Thus, $V_2 J/\mathrm{Soc}(V_2) = S \oplus S \oplus S_1$ by (3.4), and therefore the Loewy and socle structure of V_2 is as desired. $\square$

4.7. Lemma. *Let V be the trivial source kG-module in* (4.6) *such that*

$$V = \begin{matrix} T \\ SSS_1 \\ T \end{matrix} \quad \text{(Loewy and socle series)} \leftrightarrow \chi_{52668} + \chi_{58311_1}.$$

(i) *It holds that V has D as its vertex.*

(ii) $V\downarrow_H \cdot 1_B = \tilde{\varphi}_5 \oplus (46 \times P(\tilde{\varphi}_5) \oplus 41 \times P(\tilde{\varphi}_6) \oplus 41 \times P(\tilde{\varphi}_{13}) \oplus 41 \times P(\tilde{\varphi}_{14}) \oplus 72 \times P(\tilde{\varphi}_{18}))$, *so that $fV = \tilde{\varphi}_5$ and $g\tilde{\varphi}_5 = V$.*

(iii) $\tilde{\varphi}_5 \uparrow^G \cdot 1_A = V \oplus (6 \times P(S) \oplus 15 \times P(T) \oplus 3 \times P(S_1))$.

Proof. (i) Since $v_3(\dim_k V) = 2 = v_3(|G|) - v_3(|D|)$, (1.8) implies (i).

(ii) and (iii): Clearly, $\dim_k V \equiv 9 \pmod{81}$. Hence, by (i), (4.4) and (4.5)(ii), we get $V \in \{g\tilde{\varphi}_i \mid i = 5, 13, 14\}$.

Suppose that $V = g\tilde{\varphi}_{13}$. Then, $(\tilde{\varphi}_{14})\uparrow^G = (\tilde{\varphi}_{13}^{\sigma})\uparrow^G = (\tilde{\varphi}_{13}\uparrow^G)^{\sigma}$ by (2.11)(vi). Therefore, $V^{\sigma} = g\tilde{\varphi}_{14}$. On the other hand, since the maximal subgroup M of G is unique up to G-conjugacy by [6, p. 132] (or [40,41]), we know that $(k_M\uparrow^G\cdot 1_A)^\sigma\cong k_M\uparrow^G\cdot 1_A$, so that $V^\sigma\cong V$, a contradiction.

Similarly, we have a contradiction if $V=g\tilde{\varphi}_{14}$ since $\tilde{\varphi}_{13}^\sigma=\tilde{\varphi}_{14}$ by (2.11)(vi).

Thus, $V=g\tilde{\varphi}_5$. Then, the rest is easy by (4.3)(ii), (2.9)(iii), (2.11)(ii) and (2.6)(iv). □

4.8. Lemma. *Let L be the same as in* (2.13), *and let $U=k_L\uparrow^G\cdot 1_A$. Let Q be a vertex of U* (*since U is indecomposable by* (3.2)).

(i) *$U=U(S,T,S)\leftrightarrow \chi_{10944}+\chi_{52668}$ and Q is the cyclic group of order* 3.

(ii) *The Green correspondent fU of U satisfies*

$$U\downarrow_H\cdot 1_B=fU\oplus(27\times P(\tilde{\varphi}_5)\oplus 25\times P(\tilde{\varphi}_6)\oplus 24\times P(\tilde{\varphi}_{13})\oplus 24\times P(\tilde{\varphi}_{14})\oplus 40\times P(\tilde{\varphi}_{18})),$$

$$(fU)\uparrow^G\cdot 1_A=U\oplus(54\times P(S)\oplus 175\times P(T)\oplus 24\times P(S_1)\oplus 20\times P(S_2)\oplus 20\times P(S_3))$$

and fU has Loewy and socle series

$$fU=\begin{array}{c}\tilde{\varphi}_5\tilde{\varphi}_6\tilde{\varphi}_{13}\tilde{\varphi}_{14}\\ \tilde{\varphi}_{18}\tilde{\varphi}_{18}\\ \tilde{\varphi}_5\tilde{\varphi}_6\tilde{\varphi}_{13}\tilde{\varphi}_{14}\end{array}\leftrightarrow\tilde{\chi}_5+\tilde{\chi}_6+\tilde{\chi}_{13}+\tilde{\chi}_{14}+\tilde{\chi}_{25}.$$

(iii) *$U=P_Q(S)$.*

Proof. (i) By (3.2), it is sufficient to show $Q=C_3$. Clearly, $\dim_k U\equiv 27 \pmod{81}$, so that U is not projective by Notation (2.1). We can choose Q as subgroup of D. If $Q=D$, then U is a trivial source kG-module in A with vertex D, a contradiction by (4.5)(i)–(ii).

(ii) By (4.3)(ii), $U\downarrow_H\cdot 1_B=fU\oplus\tilde{P}$ for a projective kH-module $\tilde{P}$ in B. Then, fU is a trivial source kH-module in B with vertex Q, see [26, Chapter 4, Theorem 4.3(i)(c)]. Then, by (2.14), $fU=X$ or $fU=X'$, where X and X' are the modules appearing in (2.14) in this order. It follows from (2.7)(iii), [1, p. 90, Corollary 5], (4.4)(i) and [22, II, Corollary 2.8] that $0\neq(S,U)^G=\underline{\mathrm{Hom}}_{kG}(S,U)\cong\underline{\mathrm{Hom}}_{kH}(fS,fU)=\underline{\mathrm{Hom}}_{kH}(\tilde{\varphi}_6,fU)=(\tilde{\varphi}_6,fU)^H$, so that $\tilde{\varphi}_6|\mathrm{Soc}(fU)$. Thus, $fU=X$. Hence we know the Loewy and socle series of fU, and we know also the decomposition of $\chi_{\widehat{fU}}$ from (2.14)(i). Then, the rest in (ii) is easily obtained by (2.9)(iii) and (2.6)(iv).

(iii) Since U is a trivial source kG-module in A, we get by (1.7) that fU is a trivial source kH-module in B with vertex Q. Clearly, $\tilde{\varphi}_6|L_1(fU)$ by (4.4)(i). Let X be the same as in (2.14). Hence, $fU=X$ by (2.14). On the other hand, we can define $f(P_Q(S))$ by (2.7)(iii). By (4.4)(i), we know $\tilde{\varphi}_6|L_1(f(P_Q(S)))$, so that $f(P_Q(S))=X$ by (2.14). This means that $U=P_Q(S)$. □

4.9. Lemma. *We get the following*:

(i) $T\downarrow_H \cdot 1_B = fT \oplus (15\times P(\tilde{\varphi}_5) \oplus 15\times P(\tilde{\varphi}_6) \oplus 14\times P(\tilde{\varphi}_{13}) \oplus 14\times P(\tilde{\varphi}_{14}) \oplus 28\times P(\tilde{\varphi}_{18}))$, *and* fT *has Loewy and socle series*

$$fT = \begin{array}{c} \tilde{\varphi}_{13}\tilde{\varphi}_5\tilde{\varphi}_{14} \\ \tilde{\varphi}_{18}\tilde{\varphi}_{18} \\ \tilde{\varphi}_{14}\tilde{\varphi}_5\tilde{\varphi}_{13} \end{array}$$

(ii) $(fT)\uparrow^G \cdot 1_A = T \oplus (44\times P(S) \oplus 145\times P(T) \oplus 22\times P(S_1) \oplus 16\times P(S_2) \oplus 16\times P(S_3))$.

Proof. We get $[P(\tilde{\varphi}_5)|T\downarrow_H] = [P(\tilde{\varphi}_6)|T\downarrow_H] = 15$ by (4.7)(iii), and (4.4)(ii) and (1.6). Hence, $T\downarrow_H \cdot 1_B = fT \oplus (15\times P(\tilde{\varphi}_5) \oplus 15\times P(\tilde{\varphi}_6) \oplus m_{13}\times P(\tilde{\varphi}_{13}) \oplus m_{14}\times P(\tilde{\varphi}_{14}) \oplus m_{18}\times P(\tilde{\varphi}_{18}))$ for non-negative integers m_{13}, m_{14} and m_{18} by (2.6)(ii) and (4.3)(i). Hence, (4.4)(i) implies that

$$\begin{array}{lll} S\downarrow_H \cdot 1_B & & \\ & \left| T\downarrow_H \cdot 1_B \right| & \\ & & S\downarrow_H \cdot 1_B \end{array}$$

$$= \begin{pmatrix} \tilde{\varphi}_6 & & \\ & \left| fT \right| & \\ & & \tilde{\varphi}_6 \end{pmatrix}$$

$$\oplus \begin{pmatrix} 27\times P(\tilde{\varphi}_5) & \oplus 25\times P(\tilde{\varphi}_6) & \oplus (10+m_{13})\times P(\tilde{\varphi}_{13}) \\ & \oplus (10+m_{14})\times P(\tilde{\varphi}_{14}) & \oplus (12+m_{18})\times P(\tilde{\varphi}_{18}) \end{pmatrix}.$$

Let U be the same as in (4.8). Then, by (4.8)(ii),

$$fU \oplus (14\times P(\tilde{\varphi}_{13}) \oplus 14\times P(\tilde{\varphi}_{14}) \oplus 28\times P(\tilde{\varphi}_{18}))$$

$$= \begin{pmatrix} \tilde{\varphi}_6 & & \\ & \left| fT \right| & \\ & & \tilde{\varphi}_6 \end{pmatrix} \oplus (m_{13}\times P(\tilde{\varphi}_{13}) \oplus m_{14}\times P(\tilde{\varphi}_{14}) \oplus m_{18}\times P(\tilde{\varphi}_{18})).$$

Then, $m_{13} = m_{14} = 14$ and $m_{18} = 28$, and fU has Loewy and socle series as desired by (4.8)(ii).

(ii) Since $(fT)\uparrow^G \cdot 1_A = T \oplus (\text{proj})$ by (2.6)(ii) and (4.3)(ii), we get the assertion by (2.6)(iv). $\square$

4.10. Lemma. (i) $[fS_1, \tilde{\varphi}_i]^H = [\tilde{\varphi}_i, fS_1]^H = 0$ *for* $i = 5, 6$.

(ii) $[P(\tilde{\varphi}_{18})|S_1\downarrow_H] = [P(S_1)|\tilde{\varphi}_{18}\uparrow^G] = 4$.

Proof. By (2.6)(ii), let $\tilde{M} = fS_1$. Then, by (4.3)(i), $S_1\downarrow_H \cdot 1_B = \tilde{M} \oplus \tilde{Q}$ for a projective kH-module $\tilde{Q}$ in B.

(i) It follows $[\tilde{M}, \tilde{\varphi}_5]^H = [S_1\downarrow_H, \tilde{\varphi}_5]^H - [P(\tilde{\varphi}_5)|S_1\downarrow_H] = 0$ by (4.7)(iii) and (1.6). Similarly, by (4.4)(ii) and (1.6), we know $[\tilde{M}, \tilde{\varphi}_6]^H = 0$.

(ii) Let V be the same as in (4.7). Then, we get from (4.9)(i) and (4.4)(i) that

$$fT = \begin{matrix} \tilde{\varphi}_{13}\tilde{\varphi}_5\tilde{\varphi}_{14} \\ \tilde{\varphi}_{18}\tilde{\varphi}_{18} \\ \tilde{\varphi}_{14}\tilde{\varphi}_5\tilde{\varphi}_{13} \end{matrix},$$

$$V\downarrow_H \cdot 1_B = \tilde{X} \oplus \begin{pmatrix} 42 \times P(\tilde{\varphi}_5) & \oplus 40 \times P(\tilde{\varphi}_6) & \oplus 38 \times P(\tilde{\varphi}_{13}) \\ \oplus\ 38 \times P(\tilde{\varphi}_{14}) & \oplus 68 \times P(\tilde{\varphi}_{18}) & \oplus \tilde{Q} \end{pmatrix}$$

for a kH-module $\tilde{X}$ in B such that

$$\tilde{X} = \begin{matrix} fT & & \\ & \left|\tilde{\varphi}_6 \oplus \tilde{\varphi}_6 \oplus \tilde{M}\right| & \\ & & fT \end{matrix}.$$

By (1.6), (4.7)(iii) and (4.4)(ii), we can write that $\tilde{Q} = 3 \times P(\tilde{\varphi}_5) \oplus P(\tilde{\varphi}_6) \oplus m_{13} \times P(\tilde{\varphi}_{13}) \oplus m_{14} \times P(\tilde{\varphi}_{14}) \oplus m_{18} \times P(\tilde{\varphi}_{18})$ for non-negative integers m_{13}, m_{14} and m_{18}. So, (4.7)(ii) and the theorem of Krull–Schmidt imply that $\tilde{\varphi}_5 \oplus (P(\tilde{\varphi}_5) \oplus 3 \times P(\tilde{\varphi}_{13}) \oplus 3 \times P(\tilde{\varphi}_{14}) \oplus 4 \times P(\tilde{\varphi}_{18})) = \tilde{X} \oplus (m_{13} \times P(\tilde{\varphi}_{13}) \oplus m_{14} \times P(\tilde{\varphi}_{14}) \oplus m_{18} \times P(\tilde{\varphi}_{18}))$.

Now, we want to claim that $P(\tilde{\varphi}_{18}) \nmid \tilde{X}$. Let $\tilde{P} = P(\tilde{\varphi}_{18})$.

Suppose $\tilde{P} | \tilde{X}$. There are kH-submodules $\tilde{Y}$ and $\tilde{Z}$ of $\tilde{X}$ such that $\tilde{X} \supseteq \tilde{Y} \supseteq \tilde{Z}$, $\tilde{X}/\tilde{Y} \cong \tilde{Z} \cong fT$ and $\tilde{Y}/\tilde{Z} \cong \tilde{\varphi}_6 \oplus \tilde{\varphi}_6 \oplus \tilde{M}$. Since $\tilde{\varphi}_{18} \nmid \mathrm{Soc}(\tilde{Z})$ by (4.9)(i), there is a direct sum $\tilde{P} \oplus \tilde{Z} \subseteq \tilde{X}$. Thus,

$$\tilde{P} \cong (\tilde{P} \oplus \tilde{Z})/\tilde{Z} \subseteq \tilde{X}/\tilde{Z} = \begin{matrix} (\tilde{X}/\tilde{Y}) & \\ & \Big| (\tilde{Y}/\tilde{Z}) \end{matrix}.$$

Let $\tilde{U} = (\tilde{X}/\tilde{Z})/(\tilde{\varphi}_6 \oplus \tilde{\varphi}_6)$. Since $\tilde{P} \cap (\tilde{\varphi}_6 \oplus \tilde{\varphi}_6) = 0$, we similarly know that

$$\tilde{P} \cong (\tilde{P} \oplus \tilde{\varphi}_6 \oplus \tilde{\varphi}_6)/(\tilde{\varphi}_6 \oplus \tilde{\varphi}_6) \subseteq \tilde{U} = \begin{matrix} fT & \\ & \Big| \tilde{M} \end{matrix}.$$

Hence, $\tilde{U}$ has a submodule $\tilde{V}$ such that $\tilde{U}/\tilde{V} \cong \tilde{\varphi}_5 \oplus \tilde{\varphi}_{13} \oplus \tilde{\varphi}_{14}$ and $\tilde{V} = \begin{matrix} \tilde{\varphi}_{18}\tilde{\varphi}_{18} \\ \tilde{\varphi}_5\tilde{\varphi}_{13}\tilde{\varphi}_{14} \end{matrix} \Big| \tilde{M}$.

It follows from (2.11)(iv) and [22, I, Lemma 8.5(ii)] that $\tilde{P}$ has a submodule $\tilde{W}$ such that $j(\tilde{W}) = 4$ and $L_1(\tilde{W}) = \tilde{\varphi}_6$. Then, $\tilde{W} \subseteq \tilde{M}$. Since $j(\tilde{M}) \leqslant 4$ by (2.11)(iv), we finally know that $j(\tilde{M}) = 4$ and $\tilde{\varphi}_6 | L_1(\tilde{M})$, contradicting (i).

Hence, $m_{18} = 4$, which implies the rest of (ii) by (1.6). $\square$

4.11. Lemma. *It holds the following*:

(i)

$$g\tilde{\varphi}_{18} = \begin{matrix} S_1S_2S_3 \\ TT \\ S_1S_2S_3 \end{matrix} \quad \text{(Loewy and socle series)} \leftrightarrow 2 \times \chi_{58653}$$

and $\tilde{\varphi}_{18} \uparrow^G \cdot 1_A = g\tilde{\varphi}_{18} \oplus (6 \times P(S) \oplus 28 \times P(T) \oplus 4 \times P(S_1) \oplus 4 \times P(S_2) \oplus 4 \times P(S_3))$.

(ii) $(g\tilde{\varphi}_{18})\downarrow_H \cdot 1_B = \tilde{\varphi}_{18} \oplus \begin{pmatrix} 36 \times P(\tilde{\varphi}_5) \ \oplus 40 \times P(\tilde{\varphi}_6) \ \oplus 40 \times P(\tilde{\varphi}_{13}) \\ \oplus 40 \times P(\tilde{\varphi}_{14}) \ \oplus 83 \times P(\tilde{\varphi}_{18}) \end{pmatrix}$.

(iii) $[fS_1, \tilde{\varphi}_{18}]^H = [\tilde{\varphi}_{18}, fS_1]^H = 1$.

Proof. Let $X = g\tilde{\varphi}_{18}$. Note that X is a trivial source kG-module in A by (4.5)(i).

(i) By (4.3)(ii), we get $\tilde{\varphi}_{18}\uparrow^G \cdot 1_A = X \oplus P$ for a projective kG-module P in A. By (1.6), (4.4)(i), (4.9)(i), (4.10)(ii), (2.6)(iii), (2.9)(ii) and (2.11)(ii), it follows

$$P = 6 \times P(S) \oplus 28 \times P(T) \oplus 4 \times P(S_1) \oplus m \times P(S_2) \oplus m \times P(S_3) \tag{11}$$

for a non-negative integer m. Now, we get from (2.11)(ii), (2.9)(iii), (2.6)(iv) and (1.1)(i) that

$$X \ \leftrightarrow \ (4-m) \times \chi_{583112} + (4-m) \times \chi_{583113} + (10-2m) \times \chi_{58653}. \tag{12}$$

We know $[P(S)|\tilde{\varphi}_{18}\uparrow^G\,] = [P(\tilde{\varphi}_{18})|S\downarrow_H\,] = 6 = [\tilde{\varphi}_{18}, S\downarrow_H\,]^H$ by (1.6) and (4.4)(i), which means that $[X,S]^G = [S,X]^G = 0$ by the self-dualities. Similarly, $[P(T)|\tilde{\varphi}_{18}\uparrow^G\,] = [P(\tilde{\varphi}_{18})|T\downarrow_H\,] = 28 = [\tilde{\varphi}_{18}, T\downarrow_H\,]^H$ by (1.6) and (4.9)(i), and hence $[X,T]^G = [T,M]^G = 0$ by the self-dualities. That is

$$S \nmid L_1(X), \quad T \nmid L_1(X), \quad S \nmid \mathrm{Soc}(X), \quad T \nmid \mathrm{Soc}(X). \tag{13}$$

By (12) and (2.6)(iv), $m \leqslant 4$ and $c_X(T) = 18 - 4m \geqslant 2$. It follows from (2.6)(i) and (13) that $L_1(X) = \bigoplus_{i=1}^{3} a_i \times S_i$ for integers $a_i \geqslant 0$. Hence $j(X) \leqslant 4$ by (3.7) since X is non-projective indecomposable. Clearly, $j(X) \neq 1$ since $c_X(T) > 0$ as above. Since $T \nmid \mathrm{Soc}(X)$ by (13), we know that $j(X) \neq 2$ and $\neq 4$ from (3.7), and therefore $j(X)=3$. By (3.7), $L_2(X) = a \times T$ for an integer $a \geqslant 1$. Then, since $S \nmid \mathrm{Soc}(X)$ by (13) and since $c_X(S) = 8 - 2m$ by (12) and (2.6)(iv), it holds that $m = 4$.

So that (12) implies that $X \leftrightarrow 2 \times \chi_{58653}$. Hence, by (1.1)(ii) and (2.6)(iv),

$$[X,X]^G = 4 \quad \text{and} \quad X = 2 \times (S_1 + S_2 + S_3 + T), \text{ as composition factors.} \tag{14}$$

If $(S_i \oplus S_i)|L_1(X)$, then we get $(S_i \oplus S_i)|\mathrm{Soc}(X)$, so that $[X,X] \geqslant 5$, contradicting (14). Hence,

$$[S_i, \mathrm{Soc}(X)]^G = [L_1(X), S_i]^G \leqslant 1 \quad \text{for } i = 1,2,3. \tag{15}$$

If $j(X) = 4$, then we know from (13) and (14) that $\{2 \times (S_1 \oplus S_2 \oplus S_3)\}|\mathrm{Soc}(X)$, so that $X/\mathrm{Soc}_3(X) = T \oplus \cdots \oplus T$ by (3.1), contradicting (13). Hence, $j(X)=3$ by (13) and (3.1). Hence we know from (13), (14) and (15) that X has Loewy and socle series as desired.

(ii) Easy by (2.6)(ii), (4.3)(i), (i), (2.9)(iii) and (2.11)(ii).

(iii) By (4.10)(ii), $[P(\tilde{\varphi}_{18})|S_1\downarrow_H\,] = 4$. By (i), $[S_1\downarrow_H, \tilde{\varphi}_{18}]^H = [S_1, \tilde{\varphi}_{18}\uparrow^G\,]^G = 5$. Hence, $[fS_1, \tilde{\varphi}_{18}]^H = 1$ by (4.3)(i). Then, $[\tilde{\varphi}_{18}, fS_1]^H = 1$. □

4.12. Lemma. *We get the following*:

(i) $S_1 \downarrow_H \cdot 1_B = fS_1 \oplus (3 \times P(\tilde{\varphi}_5) \oplus P(\tilde{\varphi}_6) \oplus 2 \times P(\tilde{\varphi}_{13}) \oplus 2 \times P(\tilde{\varphi}_{14}) \oplus 4 \times P(\tilde{\varphi}_{18}))$ *and* fS_1 *has Loewy and socle series* $\begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_6\tilde{\varphi}_5 \\ \tilde{\varphi}_{18} \end{matrix}$.

(ii) $(fS_1) \uparrow^G \cdot 1_A = S_1 \oplus (23 \times P(S) \oplus 86 \times P(T) \oplus 13 \times P(S_1) \oplus 11 \times P(S_2) \oplus 11 \times P(S_3))$.

Proof. Let $\tilde{M} = fS_1$. (i) By (4.3)(i), $S_1 \downarrow_H \cdot 1_B = \tilde{M} \oplus \tilde{P}$ for a projective kH-module $\tilde{P}$ in B. As usual, we get from (1.6), (4.7)(iii), (4.4)(ii), (4.11)(ii), (2.7)(ii), (2.9)(ii), (2.11)(ii) and (2.6)(iii) that $\tilde{P} = 3 \times P(\tilde{\varphi}_5) \oplus P(\tilde{\varphi}_6) \oplus m \times P(\tilde{\varphi}_{13}) \oplus m \times P(\tilde{\varphi}_{14}) \oplus 4 \times P(\tilde{\varphi}_{18})$ for a non-negative integer m. Then, we know from (2.6)(iv) and (2.9)(iii) that

$$c_{\tilde{M}}(\tilde{\varphi}_i) = \begin{cases} 5 - 2m & \text{for } i = 5, 6, \\ 6 - 3m & \text{for } i = 13, 14, \\ 10 - 4m & \text{for } i = 18. \end{cases} \tag{16}$$

Hence, $m \in \{0, 1, 2\}$, which implies $c_{\tilde{M}}(\tilde{\varphi}_5) \geqslant 1$. Note by (4.10)(i) and (4.11)(iii) that

$$[\tilde{\varphi}_i, \tilde{M}]^H = [\tilde{M}, \tilde{\varphi}_i]^H = \begin{cases} 0 & \text{for } i = 5, 6, \\ 1 & \text{for } i = 18. \end{cases} \tag{17}$$

It holds $j(\tilde{M}) \leqslant 4$ by (2.11)(iv). These imply $j(\tilde{M}) = 3$ or 4.

Suppose $m = 0$ or 1. Then, $c_{\tilde{M}}(\tilde{\varphi}_5) \geqslant 3$. Let $\tilde{N} = \tilde{M}\tilde{J}^2$. By making use of (17) and (2.11)(iv), we get $c_{\tilde{M}/\tilde{N}}(\tilde{\varphi}_5) \leqslant 1$, and therefore $c_{\tilde{N}}(\tilde{\varphi}_5) \geqslant 2$. Then, by (2.11)(iv) and (17) again, $j(\tilde{M}) = 4$ and $[\tilde{\varphi}_{18}, \tilde{N}]^H \leqslant 1$. Hence, $\tilde{\varphi}_5 | L_4(\tilde{M}) | \mathrm{Soc}(\tilde{M})$, contradicting (17).

Therefore $m = 2$. If $j(\tilde{M}) = 4$, then (16) and (17) imply that $\tilde{M} = U(\tilde{\varphi}_{18}, \tilde{\varphi}_i, \tilde{\varphi}_j, \tilde{\varphi}_{18})$ for $\{i, j\} = \{5, 6\}$, contradicting (2.11)(iv). Hence, $j(\tilde{M}) = 3$, which means that $\tilde{M}$ has Loewy and socle series as desired.

(ii) Clearly, by (i) and (4.4)(i), $[P(S)|\tilde{M}\uparrow^G] = [\tilde{M}\uparrow^G, S]^G = 23$. By (i), $[P(S_1)|\tilde{M} \uparrow^G] = [\tilde{M} \uparrow^G, S_1]^G - 1 = 13$ by (i). Therefore, it follows from (2.11)(ii), (2.9)(iii) and (2.6)(v) that $[P(T)|\tilde{M} \uparrow^G] = 86$ and $[P(S_i)|\tilde{M} \uparrow^G] = 11$ for $i = 2, 3$. $\square$

4.13. Lemma. *We obtain the following*:

(i) $S_i \downarrow_H \cdot 1_B = fS_i \oplus (2 \times P(\tilde{\varphi}_6) \oplus 2 \times P(\tilde{\varphi}_{13}) \oplus 2 \times P(\tilde{\varphi}_{14}) \oplus 4 \times P(\tilde{\varphi}_{18}))$ *for* $i = 2, 3$, *and*

$$fS_2 = \begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_6\tilde{\varphi}_{14} \\ \tilde{\varphi}_{18} \end{matrix} \quad \text{(Loewy and socle series)} \quad \textit{and}$$

$$fS_3 = \begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_6\tilde{\varphi}_{13} \\ \tilde{\varphi}_{18} \end{matrix} \quad \text{(Loewy and socle series).}$$

(ii) $(fS_i)\uparrow^G \cdot 1_A = S_i \oplus (22 \times P(S) \oplus 85 \times P(T) \oplus 12 \times P(S_1) \oplus 13 \times P(S_2) \oplus 13 \times P(S_3))$.

Proof. Easy by (2.9)(iii), (2.11)(ii)–(iii), (1.6), (4.7)(iii), (4.4)(i)–(ii), (4.11)(i) and (2.6)(iv). □

We include the following lemma for the sake of completeness, though we do not need it for the proof of (0.2)(i).

4.14. Lemma. *It holds the following*:

(i) $\tilde{\varphi}_i \uparrow^G \cdot 1_A = g\tilde{\varphi}_i \oplus (5 \times P(S) \oplus 14 \times P(T) \oplus 2 \times P(S_1) \oplus 2 \times P(S_2) \oplus 2 \times P(S_3))$ *for* $i = 13, 14$, *and*

$$g\tilde{\varphi}_{13} = \begin{matrix} T \\ SSS_3 \\ T \end{matrix} \quad \text{(Loewy and socle series)} \quad \textit{and}$$

$$g\tilde{\varphi}_{14} = \begin{matrix} T \\ SSS_2 \\ T \end{matrix} \quad \text{(Loewy and socle series).}$$

(ii) $(g\tilde{\varphi}_i)\downarrow_H \cdot 1_B = \tilde{\varphi}_i \oplus (43 \times P(\tilde{\varphi}_5) \oplus 42 \times P(\tilde{\varphi}_6) \oplus 41 \times P(\tilde{\varphi}_{13}) \oplus 41 \times P(\tilde{\varphi}_{14}) \oplus 72 \times P(\tilde{\varphi}_{18}))$ *for* $i = 13, 14$.

Proof. These follow by (4.3)(i)–(ii), (4.5)(i), (1.1)(ii), (2.9)(iii), (4.4)(i), (1.6), (4.9)(i), (4.12)(i), (4.13)(i)–(ii), (2.6)(iv) and (3.1). □

Now, we finally get to a situation where we can prove our main result.

Proof of (0.2)(i). Recall the notation introduced in Notation (2.10). Let M_0 be a unique indecomposable direct summand of a $k[G \times H]$-module $1_A \cdot kG \cdot 1_B$ with vertex ΔD (actually, M_0 is the Green correspondent of A in $k[G \times H]$ with respect to $(G \times G, \Delta D, G \times H)$). Similarly, let N_0 be a unique indecomposable direct summand of a $k[H \times G]$-module $1_B \cdot kG \cdot 1_A$ with vertex ΔD. Then, it follows from (2.7)(iii) and [24, Theorem 4.6] by Linckelmann that a pair (M_0, N_0) induces a stable equivalence of Morita type between A and B.

Now, (2.11)(i) says that B is Puig (splendidly Morita) equivalent to B'. Recall that $\mathrm{IBr}(B) = \{\tilde{\varphi}_6, \tilde{\varphi}_5, \tilde{\varphi}_{14}, \tilde{\varphi}_{13}, \tilde{\varphi}_{18}\}$ and $\mathrm{IBr}(B') = \{k = 1_0, 1_1, 1_2, 1_3, 2\}$ by (2.11)(ii) and (2.10), respectively. We know by (4.4)(i), (4.9)(i), (4.12)(i) and (4.13)(i) that

$$fS = \tilde{\varphi}_6, \quad fT = \begin{matrix} \tilde{\varphi}_{13}\tilde{\varphi}_5\tilde{\varphi}_{14} \\ \tilde{\varphi}_{18}\tilde{\varphi}_{18} \\ \tilde{\varphi}_{14}\tilde{\varphi}_5\tilde{\varphi}_{13} \end{matrix}, \quad fS_1 = \begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_6\tilde{\varphi}_5 \\ \tilde{\varphi}_{18} \end{matrix}, \quad fS_2 = \begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_6\tilde{\varphi}_{14} \\ \tilde{\varphi}_{18} \end{matrix}, \quad fS_3 = \begin{matrix} \tilde{\varphi}_{18} \\ \tilde{\varphi}_6\tilde{\varphi}_{13} \\ \tilde{\varphi}_{18} \end{matrix}.$$

Let $G' = L_3(4)$ and $A' = B_0(kG')$. Hence we can identify D with a Sylow 3-subgroup of G', so let $H' = N_{G'}(D)$ and thus $H' = D{:}Q_8$. The calculation by Schneider [36, Theorem] (see [21, Lemma 6.6; 28, Example 4.6]) says that

$$f'(k_{G'}) = k, \quad f'(19) = \begin{matrix} 1_3\ 1_1\ 1_2 \\ 2\ 2 \\ 1_2\ 1_1\ 1_3 \end{matrix}, \quad f'(15_1) = \begin{matrix} 2 \\ k\ 1_1 \\ 2 \end{matrix},$$

$$f'(15_2) = \begin{matrix} 2 \\ k\ 1_2 \\ 2 \end{matrix}, \quad f'(15_3) = \begin{matrix} 2 \\ k\ 1_3 \\ 2 \end{matrix},$$

where $\{k_{G'}, 19, 15_1, 15_2, 15_3\} = \mathrm{IBr}(A')$ and $f' : G' \to H'$ is the Green correspondence with respect to (G', D, H'). Therefore, just by tracing a method Okuyama takes in [28, Example 4.6(1)–(10)], we finally know that there are a symmetric k-algebra B_4 and a (B, B_4)-bimodule M such that B and B_4 are stably equivalent afforded by M and that $(f(S') \otimes_B M)_{B_4} = (\text{simple}) \oplus (\text{projective})$ for all $S' \in \{S, T, S_1, S_2, S_3\} = \mathrm{IBr}(A)$. Okuyama in [28, Example 4.6(1)–(10)] says that there are a symmetric k-algebra B_4' (which is denoted by $A^{(4)}$ in [28, Example 4.6]) and a (B', B_4')-bimodule M' such that M' realizes a stable equivalence between B' and B_4' and that $(f'(T') \otimes_{B'} M')_{B_4'} = (\text{simple}) \oplus (\text{projective})$ for all $T' \in \mathrm{IBr}(A')$. Since B_4 and B_4' are Morita equivalent by the proof of [28, Example 4.6], we get by a result of Linckelmann [23, Theorem 2.1(iii)], namely by Okuyama [28, Proposition 3.1], that A and B are derived equivalent, and as byproduct, that A and A' are Morita equivalent. Therefore, by a result of the second author [21, Theorem 1.2], we know also that A is Morita equivalent to $B_0(k[L_3(q)])$ for any power q of a prime such that $q \equiv 4$ or $7 \pmod 9$. Furthermore, by reviewing the above proof and by making use of another result of Okuyama [29, Theorem 3], we conclude that A and B are, actually, splendidly equivalent, and hence $\hat{A}$ and $\hat{B}$ are splendidly equivalent as $\mathcal{O}$-algebras by a result of Rickard [33, Theorem 5.2] (see a paper of Harris [8, p. 75]). $\square$

Proof of (0.3). We keep the notation as in the Proof of (0.2)(i). It follows from (2.11)(i) that B' is isomorphic to a source algebra iBi of B as interior D-algebras, where i is a primitive idempotent of $C_B(D)$, namely, so-called a *source idempotent* of B. Therefore, we can assume $B' = iBi$. Then, we have a Puig equivalence between B and B' which is realized by a pair (Bi, iB) of bimodules, namely by a functor $F : \text{mod-}B \xrightarrow{\sim} \text{mod-}B'$ defined by $F(X) = X \cdot Bi$ for $X \in \text{mod-}B$. Thus, by looking at the Loewy structure in (2.11)(iv), $F(\tilde{\varphi}_{18}) = 2$.

Now, let ℓ and m be the elements in H as in [30, Lemma 5.2]. It follows from a result of O'Nan [30, Lemmas 5.1, 5.2] that $H = [(D : C_4) \times A_6] \cdot 2 = [(D : \langle \ell \rangle) \times A_6]\langle m \rangle$, $|\ell| = 4$, $|m| = 8$, $\ell^2 m^2 \in A_6$ and $[(\langle \ell \rangle \times A_6)\langle m \rangle]/A_6 = \langle \bar{\ell}, \bar{m} \rangle \cong Q_8$, see (2.7)(i). Let b be a block algebra of $k[C_G(D)]$ such that b is a root of B (and A). Let e be a block idempotent of kA_6 such that $kA_6 \cdot e \cong \mathrm{Mat}_9(k)$ as k-algebras. By (2.7)(i), $C_G(D) = D \times A_6$, and hence $1_b = e$. It is well-known that $1_B = 1_b$, so that we may consider that the above i is a primitive idempotent of kA_6 with $ie = i$. It follows from (2.11)(i) that

$$B = kHe = (k[(D:\langle\ell\rangle)\times A_6]\langle m\rangle)e = k[D:Q_8]\otimes_k kA_6\cdot e$$
$$= kH'\otimes_k kA_6\cdot e \cong \mathrm{Mat}_9(kH'), \qquad (*)$$

where the isomorphism is of k-algebras and of interior D-algebras. We can write $k_{D:C_4}\uparrow^{H'} = k_{H'}\oplus 1_1$. Clearly, $k_{H'}\otimes_k i\cdot kA_6$ and $1_1\otimes_k i\cdot kA_6$ are simple kH-modules in B. Namely, $\chi_{1_{H'}}\otimes\chi_9$, $\chi_{1_1}\otimes\chi_9\in\mathrm{Irr}(B)$, where χ_9 is a unique irreducible ordinary character of A_6 of degree 9 corresponding to e, and $\chi_{1_{H'}}$ and χ_{1_1} are irreducible ordinary characters of H' corresponding to simple kH'-modules $k_{H'}$ and 1_1, respectively. Recall that B and B' are Puig equivalent via (*). Since $\langle\ell\rangle\subseteq\mathrm{Ker}\,\tilde\chi_j$ for $j=5,6$, we know that $F(\tilde\varphi_j)=k_{H'}$ or 1_1 for $j=5,6$. Hence, it follows from (2.16) that we can assume that $F(\tilde\varphi_6)=k_{H'}$ and $F(\tilde\varphi_5)=1_1$. Then, we may assume that $F(\tilde\varphi_{14})=1_2$ and $F(\tilde\varphi_{13})=1_3$ by symmetry. Therefore, F sends $f(S)$ to $f'(k_{G'})$ by the proof of (0.2)(i). Moreover, we know from the proofs of (0.2)(i) and (2.15)(ii) that F sends $f(S_j)$ to $f'(15_j)$ for $j=1,2,3$.

Recall again that $\mathrm{IBr}(A)=\{S,T,S_1,S_2,S_3\}$ and $\mathrm{IBr}(A')=\{k,19,15_1,15_2,15_3\}$.

$$\begin{array}{ccc} A & & A' \\ \downarrow f & & \downarrow f' \\ B & \xrightarrow{F\,\sim} & B' \end{array}$$

Let U and Q be the same as in (4.8). By (4.8)(iii), $T=\Omega_Q(S)/S$. Now, by (2.7)(iii), we can define fU and we know that $fU=P_Q(fS)$ from the proof of (4.8)(iii), (2.14)(ii) and (4.4)(i). Hence, $f(T)=f(\Omega_Q(S)/S)=f(\Omega_Q(S))/fS=\Omega_Q(fS)/fS$ by using (4.8)(ii), (2.7)(iii) and (2.11)(iv) since $f\circ\Omega_Q=\Omega_Q\circ f$ by (2.7)(iii) as usual, see [26, Chapter 4, Theorem 4.10(ii)]. This means that $f(T)=\Omega_Q(\tilde\varphi_6)/\tilde\varphi_6$. On the other hand, it follows from [21, Lemma 6.6] that $f'(19)=\Omega_Q(k)/k$. Now, we want to claim that the functor F sends $P_Q(\tilde\varphi_6)$ to $P_Q(k)$.

Let $Y=P_Q(\tilde\varphi_6)$. Since B and B' are Puig equivalent via the functor F which is realized by taking a tensor product by the direct summand Bi of $k_{\Delta D}\uparrow^{H\times H'}$, it is easy to know that $F(Y)$ is a trivial source kH'-module with vertex Q by Mackey decomposition. Clearly, there is a kH'-epimorphism $F(Y)\twoheadrightarrow F(\tilde\varphi_6)=k_{H'}$ since there is a kH-epimorphism $Y\twoheadrightarrow\tilde\varphi_6$. Thus, just as in (2.14)(ii), we know $F(Y)=P_Q(k_{H'})$. Therefore, F sends $\Omega_Q(\tilde\varphi_6)$ to $\Omega_Q(k)$. These imply that F sends $f(T)$ to $f'(19)$.

We already know that A and B are stably equivalent of Morita type via f by (2.7)(iii) and (4.3), and that so are A' and B' via f' by Kunugi [21, Lemma 7.2(i)]. Therefore, from [23, Theorem 2.1(iii)], A and A' are Morita equivalent. Of course, we have known it already from the proof of (0.2)(i). However, the point now is that this Morita equivalence is induced by a ΔD-projective 3-permutation $k[G\times G']$-module since threw stable equivalence between A and B and that between A' and B' are induced by a ΔD-projective 3-permutation $k[G\times H]$-module and a ΔD-projective 3-permutation $k[G'\times H']$-module, respectively, and since B' is a source algebra of B, see [25, Proof of Lemma 2.3(i)]. Namely, A and A' are splendidly Morita (Puig) equivalent. Thus, the equivalence lifts to $\mathcal{O}$ by Rickard [33, Theorem 5.2] (see [8, p. 75]). □

4.15. Remark. We know Loewy and socle series of all projective indecomposable kG-modules in A by making use of Corollary (0.3) and [38, Theorem 2.2] since $L_3(4) \cong M_{21}$.

4.16. Remark. (0.2)(i) implies that there exists an isotypy between A and B by (2.7), (2.9)(iv), [4, A.2.2, p. 88; 8, Proposition 1.9].

5. A non-principal 3-block of the Higman–Sims simple group

In this section we prove that Broué's conjecture (0.1) holds for a non-principal 3-block A of the Higman–Sims simple group which has the same elementary abelian defect group D of order 9 as in Sections 2–4. It is announced by Holm [11, Section 5, p. 60] that he has proved it. However, he has not written it yet in detail, and furthermore, just by a few more calculations, we can determine the block algebra A up to Morita equivalence. Hicks almost determines the basic algebra of A in her thesis [10, Chapter 5]. But there is one unspecified β_1 which she does not determine (see [10, p. 133]), while it can be taken as -1 here. Therefore, it may be meaningful to write a proof of the above fact.

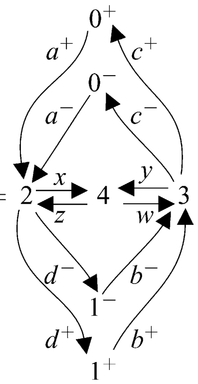
5.1. Notation and assumption. Throughout this section let $G = HS$, the Higman–Sims simple group of order $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. By Conway et al. [6, p. 80] we know the following. Namely, G has a subgroup L such that $L \cong M_{22}$. We fix a Sylow 3-subgroup D of G contained in L, which is elementary abelian of order 9. Let $H = N_G(D)$ and $H' = N_L(D)$. Then $H = (D:SD_{16}) \times C_2$, $H' = D:Q_8$, and $C_G(D) = D \times C_2$.

5.2. Notation. The group algebra kG has six blocks, namely, the principal block $B_0(kG)$, a non-principal block A with elementary abelian defect group D of order 9, a block of defect one and three blocks of defect zero. On the other hand, kH has two blocks, that is, the principal block $B_0(kH)$ and a non-principal block B such that B is the Brauer correspondent of A, and hence D is a defect group of B (see [12, Section 5, pp. 325–329]). We use the notation D, A and B throughout this section.

We use also the following notation for simple modules. Namely, $\mathrm{IBr}(A) = \{77, 49, 49^*, 154_1, 154_2, 770, 770^*\}$ where the numbers mean the k-dimensions, $\mathrm{IBr}(B) = \{0^+, 0^-, 1^+, 1^-, 2, 3 = 2^*, 4\}$ where $0^+, 0^-, 1^+, 1^-$ are of k-dimension one; and $2, 3, 4$ are of k-dimension two. Furthermore, we can write $\mathrm{IBr}(B_0(L)) = \{k_L, \widetilde{55}, \widetilde{49}, \widetilde{49}^*, \widetilde{231}\}$ and $\mathrm{IBr}(B_0(H')) = \{k_{H'}, 1_1, 1_2, 1_3, 2\}$, where the numbers mean the k-dimensions (see [12, Section 5; 38, Theorem 2.3]).

5.3. Lemma (Okuyama). *The block algebra B of kH has the following Ext-quiver with relations, and hence the projective indecomposable kH-modules in B have the following*

Loewy and socle series. 0

the Ext-*quiver of* $B =$ 

relations:

$$a^{\pm}d^{\mp}=0,\quad b^{\pm}c^{\mp}=0,\quad zx=wy,\quad a^{\pm}xz=0,\quad xzd^{\pm}=0,$$

$$b^{\pm}yw=0,\quad ywc^{\pm}=0,\quad c^{+}a^{+}=c^{-}a^{-}+yz,\quad d^{+}b^{+}=d^{-}b^{-}+xw,$$

$$c^{+}a^{+}x+c^{-}a^{-}x=0,\quad d^{+}b^{+}y+d^{-}b^{-}y=0,\quad wc^{+}a^{+}+wc^{-}a^{-}=0,$$

$$zd^{+}b^{+}+zd^{-}b^{-}=0.$$

The projective indecomposable kH-modules in B are

$$\begin{array}{c} 0^+ \\ 2 \\ 1^+\ 4 \\ 3 \\ 0^+ \end{array},\quad \begin{array}{c} 0^- \\ 2 \\ 1^-\ 4 \\ 3 \\ 0^- \end{array},\quad \begin{array}{c} 1^+ \\ 3 \\ 0^+\ 4 \\ 2 \\ 1^+ \end{array},\quad \begin{array}{c} 1^- \\ 3 \\ 0^-\ 4 \\ 2 \\ 1^- \end{array},\quad \begin{array}{c} 2 \\ 1^+\ 4\ 1^- \\ 3\ 2\ 3 \\ 0^+\ 4\ 0^- \\ 2 \end{array},\quad \begin{array}{c} 3 \\ 0^+\ 4\ 0^- \\ 2\ 3\ 2 \\ 1^+\ 4\ 1^- \\ 3 \end{array},\quad \begin{array}{c} 4 \\ 2\ 3 \\ 1^+\ 1^-\ 4\ 0^-\ 0^+ \\ 3\ 2 \\ 4 \end{array}$$

Proof. see Okuyama's calculation [28, Section 4, Case 4]. □

5.4. Lemma. *Let M_0 and N_0 be the Green correspondents of A with respect to $(G\times G, \Delta D, G\times H)$ and $(G\times G, \Delta D, H\times G)$, respectively. Then, M_0 and N_0 induce a stable equivalence of Morita type between A and B.*

Proof. By Conway et al. [6, p. 81], all elements of order 3 in G are conjugate. Let Q be a subgroup of D of order 3. Then, since $\Sigma_8 \hookrightarrow G$ and since $|C_G(Q)| = 360$ by Conway et al. [6, pp. 80–81], we easily know that $C_G(Q) = C_3 \times \Sigma_5$. Clearly, $C_H(Q) = C_3 \times \Sigma_3 \times C_2$ by Notation (5.1). Then, the non-principal 3-blocks of $k\Sigma_5$ and $k[\Sigma_3 \times C_2]$ with the same defect group C_3 are Morita (even splendidly Morita) equivalent via a pair $(M_0(\Delta D), N_0(\Delta D))$ where $M_0(\Delta D)$ is the *Brauer construction* with respect to D (see [5, p. 23; 33, p. 341]). Hence, it follows by a theorem of Broué [5, 6.3. Theorem] (cf. a proof of [33, Theorem 4.1, Lemma 4.2]) that the pair (M_0, N_0) realizes a stable equivalence of Morita type between A and B. □

5.5. Lemma. *Let $f_L : L \to H'$ and $g_L : H' \to L$ be the Green correspondences with respect to (L, D, H'). Then,*

$$g_L(k_{H'}) = k_L, \quad g_L(1_1) = \widetilde{55}, \quad g_L(1_2) = \begin{matrix} \widetilde{49}^* \\ k_L\ \widetilde{55} \\ \widetilde{49} \end{matrix},$$

$$g_L(1_3) = \begin{matrix} \widetilde{49} \\ k_L\ \widetilde{55} \\ \widetilde{49}^* \end{matrix}, \quad g_L(2) = \begin{matrix} \widetilde{231} \\ \widetilde{49}\ \widetilde{49}^* \\ \widetilde{231} \end{matrix},$$

and these five indecomposable kL-modules in $B_0(kL)$ are trivial source modules.

Proof. Easy. □

5.6. Lemma. (i) *A simple kG-module* 22 *in $B_0(kG)$ of k-dimension* 22 *is a trivial source module with vertex D.*

(ii) *Simple kG-modules* $77, 154_1, 154_2, 770$ *and* 770^* *in A are trivial source modules with vertex D.*

(iii) *There are indecomposable kG-modules X and Y such that X and Y are trivial source modules with vertex D, and that $X = U(49, 77, 49^*)$ and $Y = U(49^*, 77, 49)$.*

Proof. (i) and (ii). By Waki [39, Lemma 5.1(4)], simple kG-modules 22 and 77 are both trivial source modules. Hence [39, Lemma 5.1(5)] implies that $\widetilde{21}_{kL}$ is a trivial source module. Now, $\widetilde{21}$ belongs to a 3-block $\tilde{A}$ of kL such that a defect group of $\tilde{A}$ is C_3 and $\mathrm{IBr}(\tilde{A}) = \{\widetilde{21}, \widetilde{210}\}$ by Jansen et al. [14, p. 94]. Thus, $\widetilde{210}_{kL}$ is also a trivial source module. Therefore, by (5.5) and [39, Lemma 5.8], $770\downarrow_L$ is a 3-permutation module (a direct sum of trivial source kL-modules). Since $3 \nmid |G:L|$ by (5.1), 770 itself is a trivial source kG-module, and hence so is 770^*. Next, we get by Jansen et al. [39, Lemma 5.10] and (5.5) that $154_i\downarrow_L$ is a trivial source module for $i = 1, 2$. So, as above, 154_i is a trivial source kG-module for $i = 1, 2$.

(iii) Let X and Y be uniserial kG-modules of Loewy length 3 appearing as direct summands of $22 \otimes 154_2$ and $22 \otimes 154_1$ in [39, Lemma 5.12], respectively. Then, (i), (ii) and (1.8) imply that X and Y are our desired kG-modules. □

5.7. Lemma. *We get the following*:

(i) $154_1 \otimes_A M_0 = 0^-$, $154_2 \otimes_A M_0 = 1^-$, $770^* \otimes_A M_0 = 2$, $770 \otimes_A M_0 = 3$, $70 \otimes_A M_0 = 4$.

(ii) $49 \otimes_A M_0 = U(0^+, 2, 1^+)$, $49^* \otimes_A M_0 = U(1^+, 3, 0^+)$.

Proof. (i) Since the functor $- \otimes_A M_0$ coincides with the Green correspondence modulo projectives, it preserves the k-dimension modulo 3 and dualities. Therefore, (i) follows from (5.6)(ii).

(ii) By (5.6)(iii), we have $X \otimes_A M_0 = 0^+ \oplus (\text{proj})$, $Y \otimes_A M_0 = 1^+ \oplus (\text{proj})$. By Conway et al. [39, Theorem], a kG-module $\Omega(49)$ has a filtration

$$\Omega(49) = \begin{array}{c} 77 \\ \left| 770 \right| \\ \qquad X \end{array} .$$

Therefore, $\Omega(49 \otimes_A M_0) \equiv \Omega(49) \otimes_A M_0 \equiv U(4,3,0^+)$ (modulo proj). Hence, by (5.3), we have $49 \otimes_A M_0 = \Omega^{-1}(U(4,3,0^+)) = U(0^+,2,1^+)$. Similarly, we have $49^* \otimes_A M_0 = U(1^+,3,0^+)$. $\square$

Proof of (0.2)(ii). Let $I = \mathrm{IBr}(B)$ as in (5.2). Following Okuyama [28, Section 2(III)], take a *nice* subset I_0 of I such that $I_0 = \{0^+, 1^+\}$. Then, we know from a result of Okuyama [28, Lemma 2.1(2)] that there are a symmetric k-algebra B' which is derived equivalent to B, and a (B, B')-bimodule M' which induces a stable equivalence such that $(S \otimes_A M_0 \otimes_B M')_{B'} = (\text{simple}) \oplus (\text{proj})$ for any $S \in I$. Therefore, it follows from a result of Linckelmann [23, Theorem 2.1(iii)] that A and B' are Morita equivalent, which means that A and B are derived equivalent. Then, as before, by reviewing the above proof, we again know that this derived equivalence is actually a splendid equivalence by using Okuyama's result [29, Theorem 3]. Thus, the derived equivalence lifts from k to $\mathcal{O}$ by a result of Rickard [33, Theorem 5.2] (see a paper of Harris [8, p. 75]). $\square$

5.8. Remark. We can get (0.2)(ii) also by using a method of Rouquier [35, Theorem 7].

5.9. Lemma. *Keep the notation in the proof of* (0.2)(ii) *above. Then, the complex for B in the proof of* (0.2)(ii) *due to Okuyama is* $P(I_0)^\bullet = \bigoplus_{i \in I} P(i)^\bullet$ *such that*

$$\begin{array}{lcccccccc}
 & & -2\text{nd} & & -1\text{st} & & 0\text{th} & & 1\text{st} \\
P(0^+)^\bullet & : \cdots \to & 0 & \to & P(0^+) & \to & 0 & \to & 0 \to \cdots \\
P(0^-)^\bullet & : \cdots \to & 0 & \to & 0 & \to & P(0^-) & \to & 0 \to \cdots \\
P(1^+)^\bullet & : \cdots \to & 0 & \to & P(1^+) & \to & 0 & \to & 0 \to \cdots \\
P(1^-)^\bullet & : \cdots \to & 0 & \to & 0 & \to & P(1^-) & \to & 0 \to \cdots \\
P(2)^\bullet & : \cdots \to & 0 & \to & P(1^+) & \xrightarrow{d^+} & P(2) & \to & 0 \to \cdots \\
P(3)^\bullet & : \cdots \to & 0 & \to & P(0^+) & \xrightarrow{c^+} & P(3) & \to & 0 \to \cdots \\
P(4)^\bullet & : \cdots \to & 0 & \to & \begin{array}{c} P(0^+) \\ \oplus \\ P(1^+) \end{array} & \xrightarrow{(wc^+,\, zd^+)} & P(4) & \to & 0 \to \cdots
\end{array}$$

where $d^+ : P(1^+) \to P(2)$ *is given by* $d^+(v) = d^+ v$ *for* $v \in P(1^+) = e_{1^+}B$, *similar for* $c^+ : P(0^+) \to P(3)$, *and*

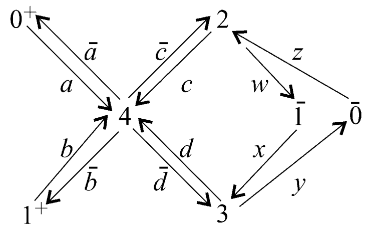
$$\begin{array}{c} P(0^+) \\ \oplus \\ P(1^+) \end{array} \xrightarrow{(wc^+,\, zd^+)} P(4)$$

is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto (wc^+,\ zd^+)\begin{pmatrix} u \\ v \end{pmatrix} = wc^+u + zd^+v \quad for \ \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix}.$$

Proof. Easy by Okuyama [28, Lemma 2.1 and its proof] and the structure of projective indecomposable B-modules in (5.3). $\square$

5.10. Theorem (see (0.4) and Hicks [10, p. 133]).
The basic algebra B' of A has the following Ext-*quiver and relations*: 0



$$a\bar{a} = 0,\ a\bar{c} = 0,\ b\bar{b} = 0,\ b\bar{d} = 0,\ c\bar{a} = 0,\ d\bar{b} = 0,$$
$$a\bar{d}y = 0,\ b\bar{c}w = 0,\ xd\bar{a} = 0,\ xd\bar{d} = 0,\ zc\bar{b} = 0,\ zc\bar{c} = 0,\ c\bar{c}w = 0,\ d\bar{d}y = 0,$$
$$-\bar{a}a + \bar{b}b + \bar{c}c - \bar{d}d = 0,$$
$$c\bar{d} = wx,\ d\bar{c} = yz,$$
$$a\bar{d}d = a\bar{b}b,\ c\bar{d}d = -c\bar{b}b,\ \bar{d}d\bar{a} = \bar{b}b\bar{a},\ \bar{d}d\bar{c} = -\bar{b}b\bar{c},$$
$$b\bar{c}c = b\bar{a}a,\ d\bar{c}c = -d\bar{a}a,\ \bar{c}c\bar{b} = \bar{a}a\bar{b},\ \bar{c}c\bar{d} = -\bar{a}a\bar{d},$$
$$c\bar{c}c = c\bar{b}b,\ d\bar{d}d = d\bar{a}a,\ \bar{c}c\bar{c} = \bar{b}b\bar{c},\ \bar{d}d\bar{d} = \bar{a}a\bar{d},$$

where the 12 *arrows here are defined by using the notation in* (5.9) *such that*

$$\begin{tikzcd} P(4)^\bullet \arrow[d, "a"'] & : & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} \arrow[r, "{(wc^+,\ zd^+)}"] \arrow[d, "{(e_{0^+}, 0)}"] & P(4) \arrow[d, "0"] \\ P(0^+)^\bullet & : & P(0^+) \arrow[r, "0"'] & 0 \end{tikzcd}
\qquad
\begin{tikzcd} P(0^+)^\bullet \arrow[d, "\bar{a}"'] & : & P(0^+) \arrow[r, "0"] \arrow[d, "{\begin{pmatrix} 0 \\ b^+c^+ \end{pmatrix}}"] & 0 \arrow[d, "0"] \\ P(4)^\bullet & : & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} \arrow[r, "{(wc^+,\ zd^+)}"'] & P(4) \end{tikzcd}$$

$$\begin{tikzcd} P(4)^\bullet \arrow[d, "b"'] & : & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} \arrow[r, "{(wc^+,\ zd^+)}"] \arrow[d, "{(0, e_{1^+})}"] & P(4) \arrow[d, "0"] \\ P(1^+)^\bullet & : & P(1^+) \arrow[r, "0"'] & 0 \end{tikzcd}
\qquad
\begin{tikzcd} P(1^+)^\bullet \arrow[d, "\bar{b}"'] & : & P(1^+) \arrow[r, "0"] \arrow[d, "{\begin{pmatrix} a^+d^+ \\ 0 \end{pmatrix}}"] & 0 \arrow[d, "0"] \\ P(4)^\bullet & : & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} \arrow[r, "{(wc^+,\ zd^+)}"'] & P(4) \end{tikzcd}$$

$$
\begin{array}{cccc}
P(4)^{\bullet}: & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} & \xrightarrow{(wc^+,\, zd^+)} & P(4) \\
c\big\downarrow & \big\downarrow (b^+c^+, 0) & & \big\downarrow x \\
P(2)^{\bullet}: & P(1^+) & \xrightarrow[d^+]{} & P(2)
\end{array}
\qquad
\begin{array}{cccc}
P(2)^{\bullet}: & P(1^+) & \xrightarrow{d^+} & 0 \\
\bar{c}\big\downarrow & \big\downarrow \begin{pmatrix} 0 \\ e_{1^+} \end{pmatrix} & & \big\downarrow z \\
P(4)^{\bullet}: & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} & \xrightarrow[(wc^+,\, zd^+)]{} & P(4)
\end{array}
$$

$$
\begin{array}{cccc}
P(4)^{\bullet}: & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} & \xrightarrow{(wc^+,\, zd^+)} & P(4) \\
d\big\downarrow & \big\downarrow (0, a^+d^+) & & \big\downarrow y \\
P(3)^{\bullet}: & P(0^+) & \xrightarrow[c^+]{} & P(3)
\end{array}
\qquad
\begin{array}{cccc}
P(3)^{\bullet}: & P(0^+) & \longrightarrow & P(3) \\
\big\downarrow \bar{d} & \big\downarrow \begin{pmatrix} c^+e_{0^+} \\ 0 \end{pmatrix} & & \big\downarrow w \\
P(4)^{\bullet}: & \begin{matrix} P(0^+) \\ \oplus \\ P(1^+) \end{matrix} & \xrightarrow[(wc^+,\, zd^+)]{} & P(4)
\end{array}
$$

$$
\begin{array}{cccc}
P(3)^{\bullet}: & P(0^+) & \xrightarrow{c^+} & P(3) \\
x\big\downarrow & \big\downarrow 0 & & \big\downarrow -b^- \\
P(1^-)^{\bullet}: & 0 & \xrightarrow[0]{} & P(1^-)
\end{array}
\qquad
\begin{array}{cccc}
P(0^-)^{\bullet}: & 0 & \xrightarrow[0]{} & P(0^-) \\
y\big\downarrow & \big\downarrow 0 & & \big\downarrow -c^- \\
P(3)^{\bullet}: & P(0^+) & \xrightarrow[c^+]{} & P(3)
\end{array}
$$

$$
\begin{array}{cccc}
P(2)^{\bullet}: & P(1^+) & \xrightarrow{d^+} & P(2) \\
z\big\downarrow & \big\downarrow 0 & & \big\downarrow a^- \\
P(0^-)^{\bullet}: & 0 & \xrightarrow[0]{} & P(0^-)
\end{array}
\qquad
\begin{array}{cccc}
P(1^-)^{\bullet}: & 0 & \xrightarrow{0} & P(1^-) \\
w\big\downarrow & \big\downarrow 0 & & \big\downarrow d^- \\
P(2)^{\bullet}: & P(1^+) & \xrightarrow[d^+]{} & P(2)
\end{array}
$$

Proof. This follows by using (5.9), the Ext-quiver of B and its relations in (5.3). □

5.11. Remark. The unspecified β_1 in [10, p. 133] is taken as $\beta_1 = -1$ by (5.10).

5.12. Remark. By (5.1), A and B satisfy [4, 4.2. Hypothèses]. Hence, (0.2)(ii) and [8, Proposition 1.9] imply that there exists an isotypy between A and B.

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